Math 312 - Quiz 1 Solutions

1. Find the gcd \((70, 105, 98)\) using the Euclidean algorithm.

   \textbf{Solution:} \(\text{gcd}(70, 105, 98) = \text{gcd}(\text{gcd}(70, 105), 98)\) (1 point). By the Euclidean algorithm, \(105 = 1 \cdot 70 + 35\) and \(70 = 2 \cdot 35 + 0\), so \(\text{gcd}(70, 105) = 35\) (2 points). Apply the Euclidean algorithm again to get \(98 = 2 \cdot 35 + 28, 35 = 1 \cdot 28 + 7\) and \(28 = 4 \cdot 7 + 0\), so \(\text{gcd}(35, 98) = 7\) (2 points).

   Any solution using unique factorization is awarded marks as well.

2. Using mathematical induction, prove that \(2^n < n!\) for \(n \geq 4\).

   \textbf{Solution:} The base case \(n = 4\) is \(16 < 24\) which is true (1 point). Assume that the statement is true for some \(n \geq 4\). Then, \(2^{n+1} = 2 \cdot 2^n < 2 \cdot n!\) by the induction hypothesis (2 points). Now \(2 < n+1\) because \(n+1 \geq 5 > 2\) (1 point). Combining these two inequalities gives \(2^{n+1} < (n+1)!\) (1 point).

3. State whether the following are true or false:

   (a) If \(p_k\) is the \(k\)-th prime, then it is asymptotic to \(\ln k\).

      \textbf{Solution:} TRUE. This statement is a version of the Prime Number Theorem. However, it is not so easy to prove that this is equivalent to \(\pi(x) \sim \frac{x}{\log x}\).

   (b) There are infinitely many primes in the set \(\{2, 9, 16, 23, 30, \ldots\}\).

      \textbf{Solution:} TRUE. This is Dirichlet’s theorem for the arithmetic progression \(7n + 2, n \geq 0\) (as 2 and 7 are coprime).

   (c) If \(d = (a, b)\) then there exists a linear combination of \(a\) and \(b\) which is strictly less than \(d\).

      \textbf{Solution:} TRUE. There is obviously \(0 = 0 \cdot a + 0 \cdot b\) which is strictly less than \(d\). The gcd is the least *positive* linear combination of \(a\) and \(b\).

      Note: If you wrote down in the quiz that you were talking about only the linear combinations which are positive, then marks were awarded this time.

   (d) If \(n\) is a positive integer, then \(\lfloor \frac{x}{n} \rfloor = \lfloor \frac{x}{n} \rfloor\).

      \textbf{Solution:} TRUE. Write \(x = \lfloor x \rfloor + \{x\}\). Then

      \[
      \left\lfloor \frac{x}{n} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor}{n} + \frac{\{x\}}{n} \right\rfloor
      \]

      Note that \(\frac{\{x\}}{n} \in [0, \frac{1}{n})\). If \(\frac{\{x\}}{n}\) is an integer, then

      \[
      \frac{\lfloor x \rfloor}{n} \leq \frac{\lfloor x \rfloor}{n} + \frac{\{x\}}{n} < \frac{\lfloor x \rfloor}{n} + 1
      \]
which shows the equality. Otherwise $\frac{x}{n}$ is a rational number which is not an integer. So $d < \frac{x}{n} < d + 1$ for some integer $d$. But since $\frac{x}{n} \in [0, \frac{1}{n})$,
\[ d < \frac{x}{n} + \frac{\{x\}}{n} < d + 1 \]
as well. This proves the equality both cases.

(e) There exist integers $(m, n)$ such that $11m + 5n = 1$.

**Solution:** TRUE. Follows from Bezout’s theorem since the gcd of 11 and 5 is 1. Alternatively, $11 \cdot 1 + 5 \cdot (-2) = 1$. 

2