1) Suppose $\sqrt[3]{5}=\frac{a}{b}$, with $a$ and $b$ integers and $(a,b)=1$. Then $5=\frac{a^3}{b^3}$, or $5b^3=a^3$. Then $5|a^3$, so $5|a$ and we have $5^3|a^3$. Then $5^3|5b^3$, or $5^2|b^3$. But then $5|b$, so $5|(a,b)$, a contradiction. Therefore $\sqrt[3]{5}$ is irrational.

2) Suppose that both 13-year and 17-year cicadas emerge in a location in 1900. The 13-year cicada will emerge again in years $1900 + 13k$ where $k$ is a positive integer. The 17-year cicadas will emerge again in years $1900 + 17k$ where $k$ is a positive integer. Both 13-year and 17-year cicadas will emerge again in years $1900 + [13, 17]k = 1900 + 221k$ where $k$ is a positive integer. Hence, they both will emerge again in the year 2121.

3) Let $x$, $y$ and $z$ be the number of turkeys, hens and chickens respectively. The problem leads to the system of Diophantine equations $x + y + z = 100$, $5x + 3y + z/3=100$. Substituting $z = 100 - x - y$ into the second equation and clearing fractions yields $14x + 8y = 200$, which has solutions $x = 4t$, $y = 25 - 7t$. It follows that $z = 75+3t$. The only values for $t$ which make all three of these numbers nonnegative are $t = 0, 1, 2, and 3$. Thus the solutions to the problem are $(x, y, z) = (0, 25, 75); (4, 18, 78); (8, 11, 81); (12, 4, 84)$.

4) Suppose that $a$ is odd. Then $a = 2k+1$ for some integer $k$. Then $a^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 4k(k+1) + 1$.

If $k$ is even, then $k=2l$ where $l$ is an integer. Then $a^2 = 8l(2l+1) + 1$. Hence $a^2 \equiv 1 \pmod{8}$. If $k$ is odd, then $k=2l+1$ when $l$ is an integer. Then $a^2 = 4(2l+1)(2l+2)+1 = 8(2l+1)(l+1)+1$. Hence $a^2 \equiv 1 \pmod{8}$. It follows that $a^2 \equiv 1 \pmod{8}$ whenever $a$ is odd.

5) a) Because $n! \equiv 0 \pmod{2}$ if $n \geq 2$, we have $1!+2!+3!+\cdots+100! \equiv 1 \pmod{2}$.

b) We have $n! \equiv 0 \pmod{7}$ whenever $n \geq 7$. Because $1! \equiv 1 \pmod{7}$, $2! \equiv 2 \pmod{7}$, $3! \equiv 6 \pmod{7}$, $4! \equiv 24 \equiv 3 \pmod{7}$, $5! \equiv 120 \equiv 1 \pmod{7}$ and $6! \equiv 720 \equiv 6 \pmod{7}$, we have $1! + 2!+3!+\cdots+100! \equiv 1!+2!+3!+4!+5!+6!\equiv 1+2+6+3+1+6=5 \pmod{7}$.

c) Because $n! \equiv 0 \pmod{12}$ whenever $n \geq 4$, it follows that
Solutions to Exercise sheet 3

1!+2!+3!+···+100! ≡ 1+2+6 ≡ 9 (mod 12).

d) Because n! ≡ 0 (mod 25) whenever n ≥ 10, it follows that

1!+2!+3!+···+100! ≡ 1!+2!+3!+ 4!+5!+6!+7!+8!+9! ≡ 1+2+6+24+20+20+15+20+5 ≡ 13 (mod 25).