Math 312 - Exercise Sheet 2 Solutions

1. (3 points) Let $a, b \in \mathbb{Z}$ such that $a$ is even and $b$ is odd. Is $\gcd(a, b)$ an even integer? Justify.

**Solution:** No. If $d = (a, b)$ was even, then $2 | d$ and $d | b$. So $2 | b$, a contradiction. Alternatively, giving an example is enough, say $a = 6, b = 9$.

2. (4 points) Let $a, b \in \mathbb{Z}$. Show that $\gcd(a^2 + b^2, a + b) = \gcd(2a^2, a + b)$.

**Solution:** We know that if $a, b, c$ are integers, then $(a + cb, b) = (a, b)$.
Using this here, $(2a^2, a + b) = (a^2 + b^2 + (a - b)(a + b), a + b) = (a^2 + b^2, a + b)$.

3. (5 points) Let $k$ be a positive integer. Use the Euclidean algorithm to show that $3k + 2$ and $5k + 3$ are relatively prime.

**Solution:** Using the Euclidean algorithm, we have the following steps:

$$
5k + 3 = 1(3k + 2) + (2k + 1)
$$

$$
3k + 2 = 1(2k + 1) + (k + 1)
$$

$$
2k + 1 = 1(k + 1) + k
$$

$$
k + 1 = 1(k) + 1
$$

So, $(5k + 3, 3k + 2) = 1$.

4. (4 points) Use the fundamental theorem of arithmetic to find the greatest common divisor and the least common multiple of 343 and 999.

**Solution:** The prime factorizations of 343 and 999 are $343 = 7^3$ and $999 = 3^3 \times 37$. So the gcd of these numbers is 1 and the lcm is $3^3 \times 7^3 \times 37 = 343 \times 999$.

5. (4 points) Use the fact that $10^2 = 100$ and $11^2 = 121$ to show that 101 is a prime number.

**Solution:** We know that any composite integer $n$ has a prime factor which is atmost $\sqrt{n}$. If 101 was composite, it would have a prime factor which is atmost $\sqrt{101} < 11$. But the primes 2, 3, 5, 7, 11 do not divide 101, a contradiction. So 101 is prime.