

2-dimensional extended Topological Quantum Field Theories and

Khovanov homology

math.AT/0510664
math.QA/0602047

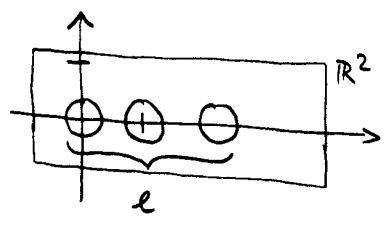
- 1 Cobordisms and Topological Quantum Field Theories (TQFTs)
- 2 Manifolds with corners and extended TQFTs
- 3 Combinatorial construction
- 4 Khovanov homology from links to tangles

1 Cobordisms and TQFTs

- try to classify cobordisms
- use vector spaces and linear maps to compute invariants
- translate from topology to algebra

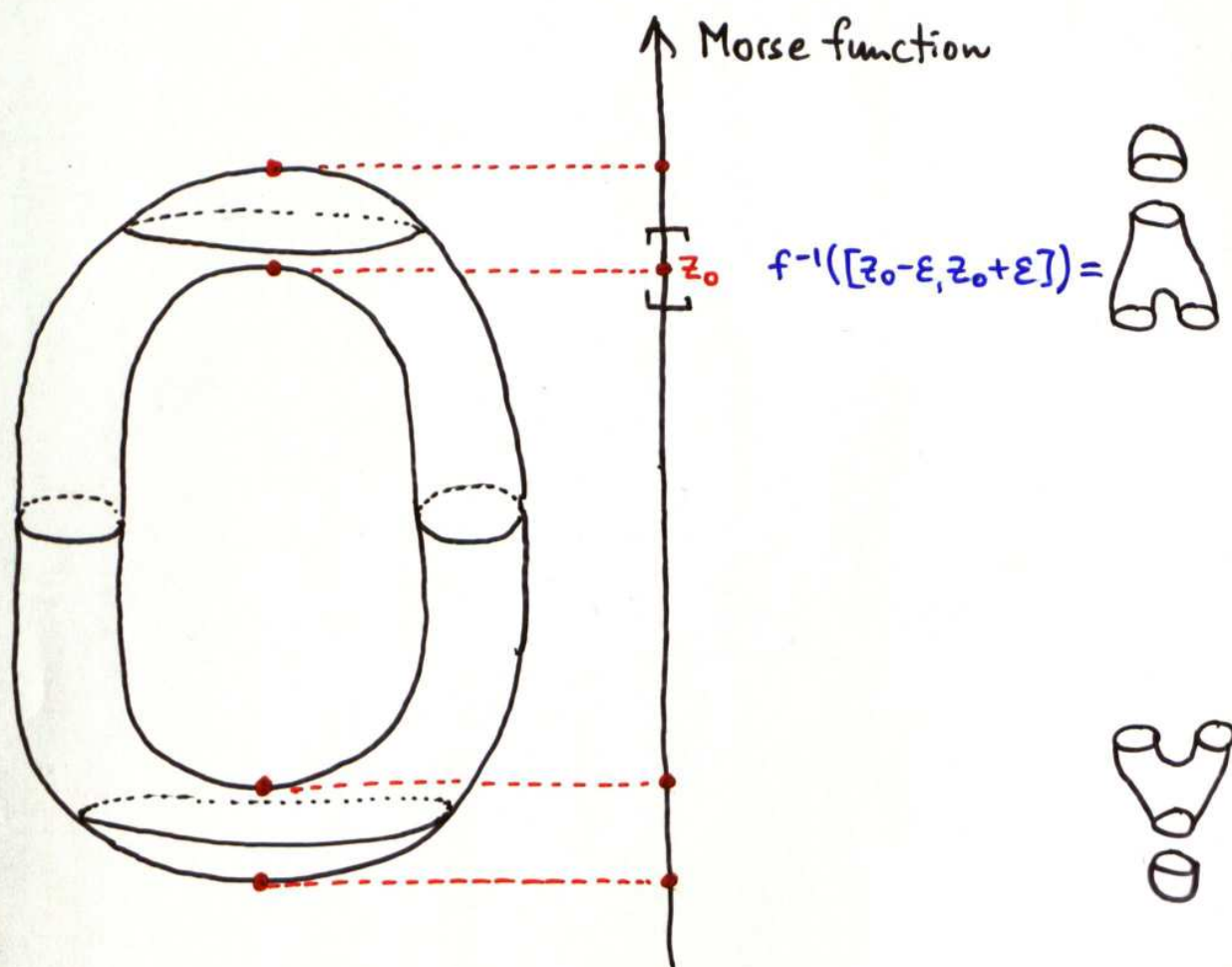
$l \in \mathbb{N}_0, \quad \Sigma_l := \bigcup_{j=0}^{l-1} \partial B_{1/4}((j, 0)) \subseteq \mathbb{R}^2$

(induced orientation)



Σ_l^* (opposite orientation)

$$T^2 \subseteq \mathbb{R}^3$$



A 2-dimensional TQFT sends

(1) closed 1-manifolds to k -vector spaces

$$\begin{array}{ccc}
 \bigcirc & \mapsto & \mathcal{C} \\
 \bigcirc \quad \bigcirc & \mapsto & \mathcal{C} \otimes \mathcal{C} \\
 \emptyset & \mapsto & k
 \end{array}$$

(2) 2-dimensional cobordisms to k -linear maps

$$\begin{array}{ccc}
 \bigcirc & \mapsto & k \xrightarrow{\eta} \mathcal{C} \\
 \text{pair of pants} & \mapsto & \mathcal{C} \otimes \mathcal{C} \xrightarrow{\mu} \mathcal{C}
 \end{array}$$

def: The category 2Cob of 2-dimensional cobordisms consists of

objects: $l \in \mathbb{N}_0$

morphisms: $[(M, \psi_1, \psi_2)] : l \rightarrow l'$

M compact oriented smooth 2-manifold

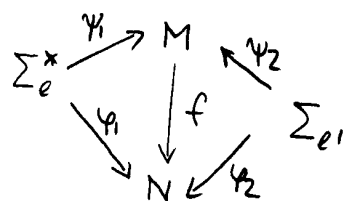
$\psi_1: \Sigma_l^* \rightarrow \psi_1(\Sigma_l^*) \subseteq \partial M$

$\psi_2: \Sigma_{l'} \rightarrow \psi_2(\Sigma_{l'}) \subseteq \partial M$

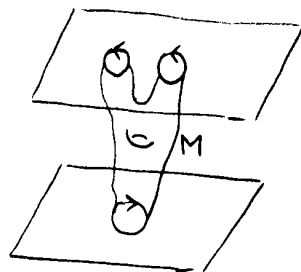
} orientation preserving diffeos

such that $\partial M = \psi_1(\Sigma_l^*) \cup \psi_2(\Sigma_{l'})$

$(M, \psi_1, \psi_2) \sim (N, \varphi_1, \varphi_2)$ iff $\exists f: M \rightarrow N$ orientation preserving diffeomorphism such that

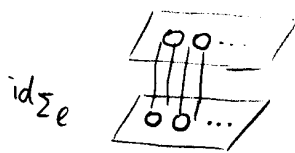


picture: Σ_2^*

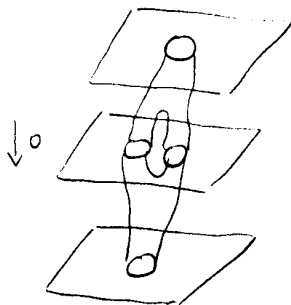


up to diffeomorphism rel ∂
(not ambient isotopy!)

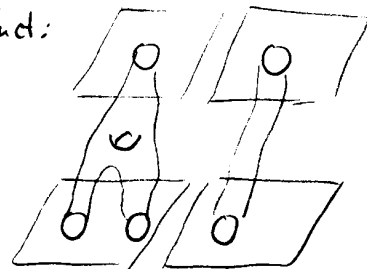
identity:



Composition:

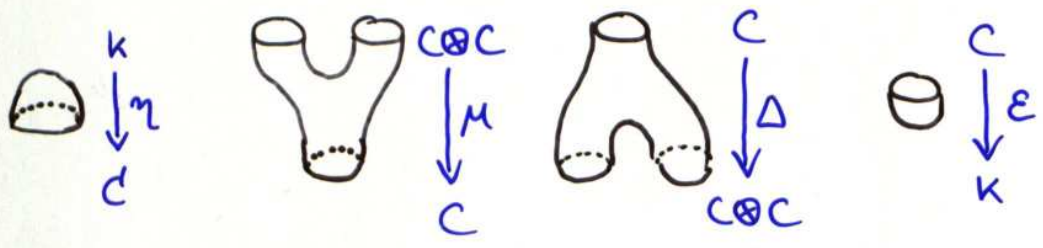


tensor product:

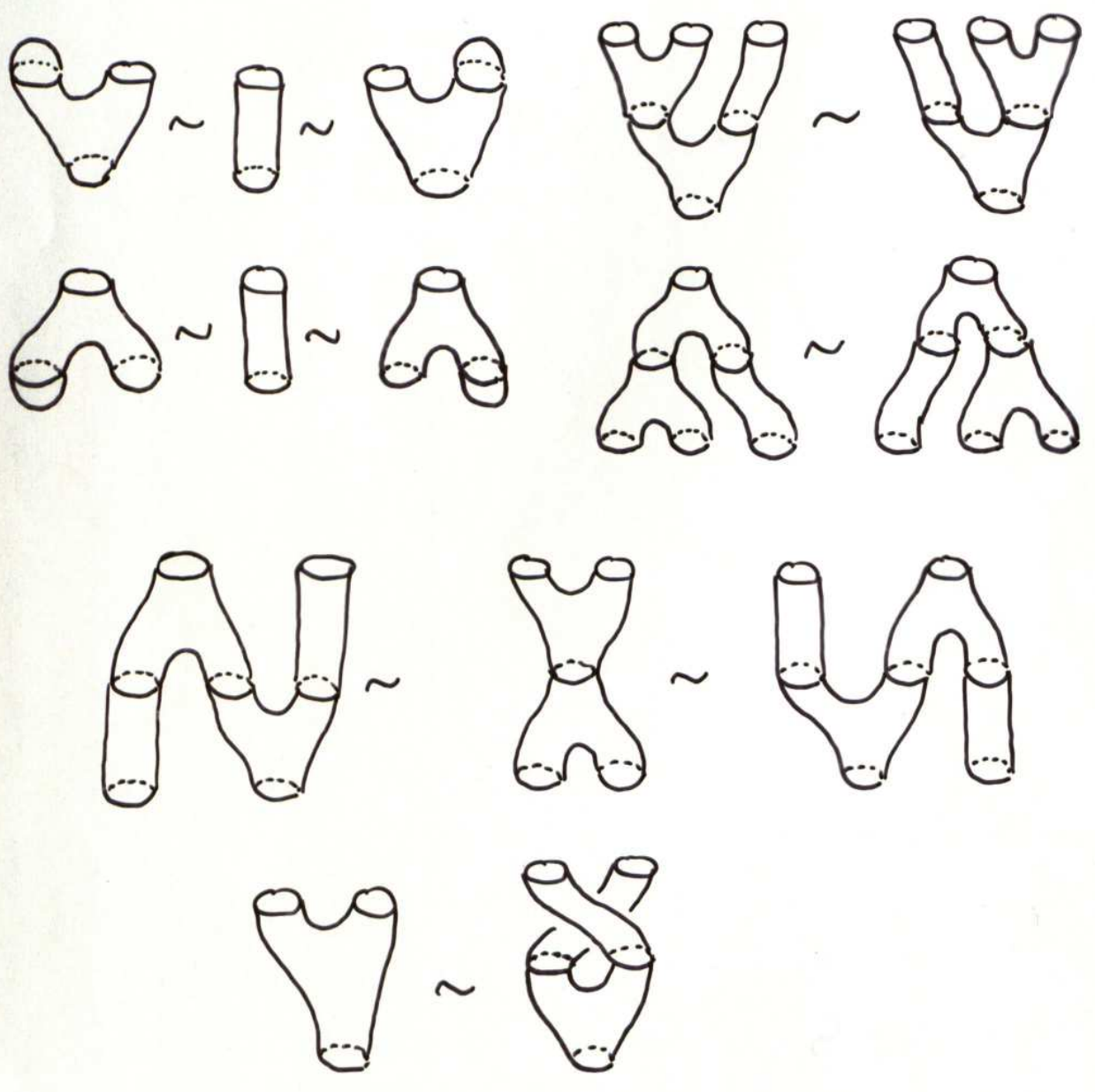


def: A 2-dimensional TQFT is a symmetric monoidal functor

$Z: 2\text{Cob} \rightarrow \text{Vect}_k$



The following morphisms of 2Cob are equivalent:



def: A Frobenius algebra $(C, \mu, \eta, \Delta, \varepsilon)$ is a k -vector space C with linear maps

$$\eta: k \rightarrow C, \quad \mu: C \otimes C \rightarrow C, \quad \Delta: C \rightarrow C \otimes C, \quad \varepsilon: C \rightarrow k$$

such that

- (a) (C, μ, η) is a unital associative algebra
- (b) (C, Δ, ε) is a comital coassociative coalgebra, i.e.
 - (i) $(id \otimes \varepsilon) \circ \Delta = id_C = (\varepsilon \otimes id) \circ \Delta$
 - (ii) $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$
- (c) $(id \otimes \mu) \circ (\Delta \otimes id) = \Delta \circ \mu = (\mu \otimes id) \circ (id \otimes \Delta)$

It is called commutative if

$$\mu \circ \tau = \mu$$

It is called symmetric if

$$\tau(a \otimes b) = b \otimes a$$

$$\varepsilon \circ \mu \circ \tau = \varepsilon \circ \mu$$

thm: [Abrams, Sawin]

The category of 2d TQFTs is equivalent to the category of commutative Frobenius algebras:

$$\text{Sym Mon Funct} (2\text{Cob}, \text{Vect}_k) \cong \text{ComFrob}$$

example:

$$C = k[x]/(x^2)$$

[Khovanov]

$$h, t \in k$$

$$C_{h,t} = k[x]/(x^2 - hx - t)$$

$$\Delta(1) = 1 \otimes x + x \otimes 1$$

$$\varepsilon(1) = 0$$

$$\Delta(1) = 1 \otimes x + x \otimes 1 - h1 \otimes 1 \quad \varepsilon(1) = 0$$

$$\Delta(x) = x \otimes x$$

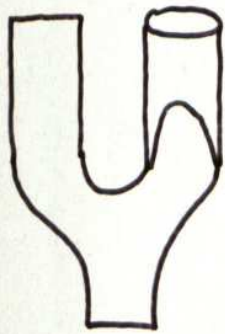
$$\varepsilon(x) = 1$$

$$\Delta(x) = x \otimes x + t1 \otimes 1 \quad \varepsilon(x) = 0$$

aim: Extend the notion of cobordisms to manifolds whose boundaries are --- 's or O 's.



A 2-dimensional $\langle 2 \rangle$ -manifold:



M



∂M



$\partial_0 M$

"black"



$\partial_1 M$

"white"

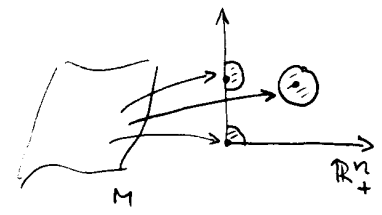


$\partial_0 M \cap \partial_1 M$

"corners"

remark: A smooth manifold with corners M has coordinate systems $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ such that the charts are homeomorphisms

$$\varphi_\alpha: U_\alpha \rightarrow \varphi_\alpha(U_\alpha) \subseteq \mathbb{R}_+^n := [0, \infty)^n$$



and transition functions

$$\varphi_\beta \circ \varphi_\alpha^{-1}: \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta)$$

are restrictions to \mathbb{R}_+^n of diffeomorphisms.

def: $p \in U_\alpha \subseteq M$

$$c(p) = \#\{i \mid (\varphi_\alpha(p))_i = 0\} \quad (\text{number of zero coefficients})$$

● A connected face is the closure of a component of $\{p \in M \mid c(p) = 1\}$

A face is a union of pairwise disjoint connected faces.

A manifold with faces is a smooth manifold with corners such that each $p \in M$ is contained in $c(p)$ different connected faces

A <2>-manifold $(M, \partial_0 M, \partial_1 M)$ is a manifold with



faces M with faces $\partial_0 M, \partial_1 M$ such that

(i) $\partial M = \partial_0 M \cup \partial_1 M$

(ii) $\partial_0 M \cap \partial_1 M$ is a face of both $\partial_0 M$ and $\partial_1 M$.

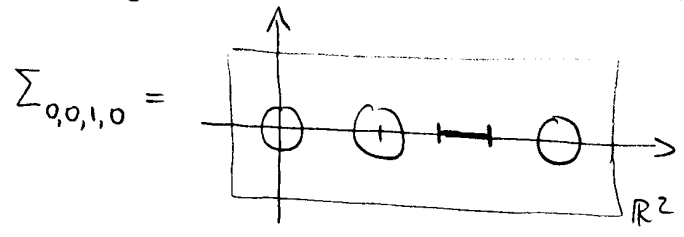
$l \in \mathbb{N}_0$

$\underline{n} = (n_0, \dots, n_{l-1}) \in \{0, 1\}^l$

$\Sigma_{\underline{n}} := \bigcup_{j=0}^{l-1} \Sigma^{(n_j)} \subseteq \mathbb{R}^2$

$\Sigma^{(0)}(x, y) := \partial B_{1/4}(x, y)$

$\Sigma^{(1)}(x, y) := [x - 1/4, x + 1/4] \times \{y\}$



def: The category 2Cob^{ext} of 2-dimensional open-closed cobordisms consists of

objects: $\underline{n} \in \{0, 1\}^l, l \in \mathbb{N}_0$

morphisms: $[(M, \psi_1, \psi_2)] : \underline{n} \rightarrow \underline{n}'$

M compact oriented 2-dimensional $\langle 2 \rangle$ -manifold

$\psi_1 : \Sigma_{\underline{n}}^* \rightarrow \psi_1(\Sigma_{\underline{n}}^*) \subseteq \partial_0 M$

$\psi_2 : \Sigma_{\underline{n}'} \rightarrow \psi_2(\Sigma_{\underline{n}'}) \subseteq \partial_0 M$

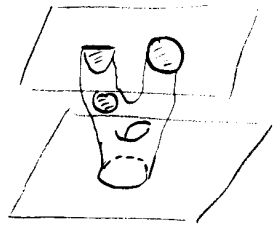
} orientation-preserving diffeo

such that

$\partial_0 M = \psi_1(\Sigma_{\underline{n}}^*) \cup \psi_2(\Sigma_{\underline{n}'})$

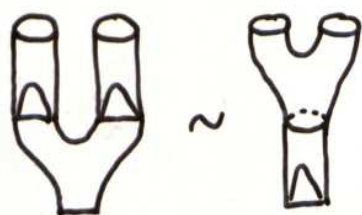
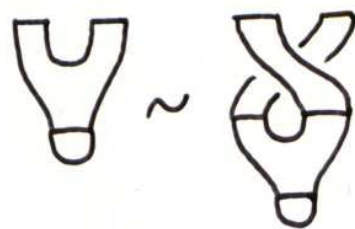
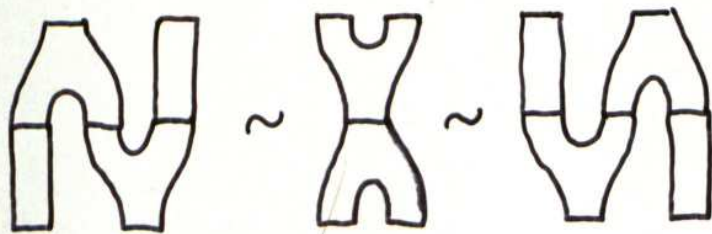
" \sim " as above.

picture:



def: A 2-dimensional open-closed TQFT is a symmetric monoidal functor

$Z : 2\text{Cob}^{\text{ext}} \rightarrow \text{Vect}_k$



~



Knowledge



~



duality



~



Cardy condition

def: A Knowledgeable Frobenius algebra (A, C, ι, ι^*) consists of

- (a) a symmetric Frobenius algebra $(A, \mu_A, \eta_A, \Delta_A, \varepsilon_A)$
- (b) a commutative Frobenius algebra $(C, \mu_C, \eta_C, \Delta_C, \varepsilon_C)$
- (c) linear maps $\iota: C \rightarrow A, \iota^*: A \rightarrow C$

such that

- (i) ι is a homomorphism of algebras
- (ii) $\mu_A \circ (\iota \otimes \text{id}_A) = \mu_A \circ \tau \circ (\iota \otimes \text{id}_A)$
- (iii) $\varepsilon_C \circ \mu_C \circ (\text{id}_C \otimes \iota^*) = \varepsilon_A \circ \mu_A \circ (\iota \otimes \text{id}_A)$
- (iv) $\iota \circ \iota^* = \mu_A \circ \tau \circ \Delta_A$

Thm: [AL, HP]

The category 2Cob^{ext} is the free ^{strict} symmetric monoidal category generated by a knowledgeable Frobenius algebra object.

Cor:

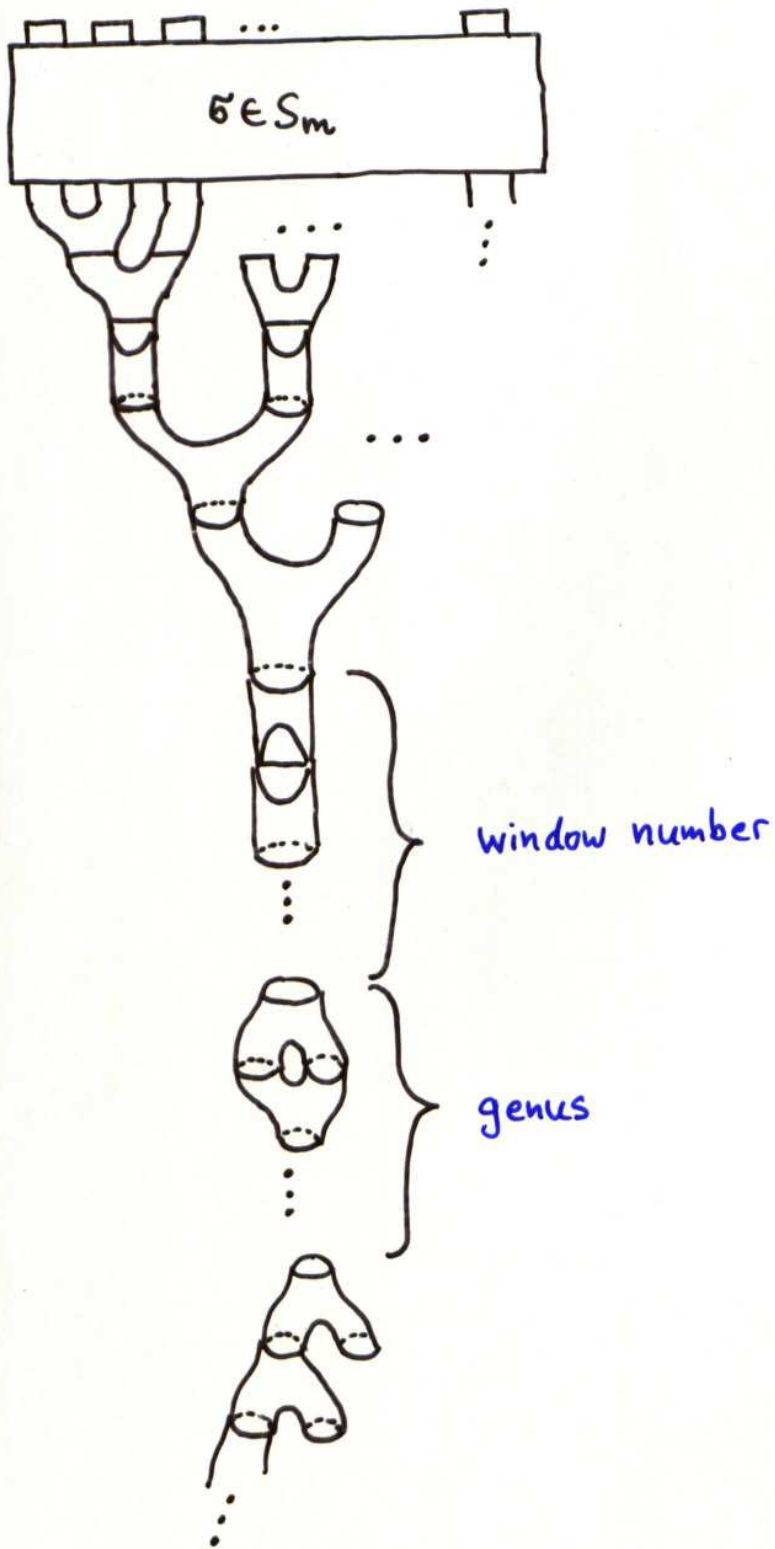
$$\text{SymMonFunct} [2\text{Cob}^{\text{ext}}, \text{Vect}_K] \cong K\text{-Frob}$$

Proof:

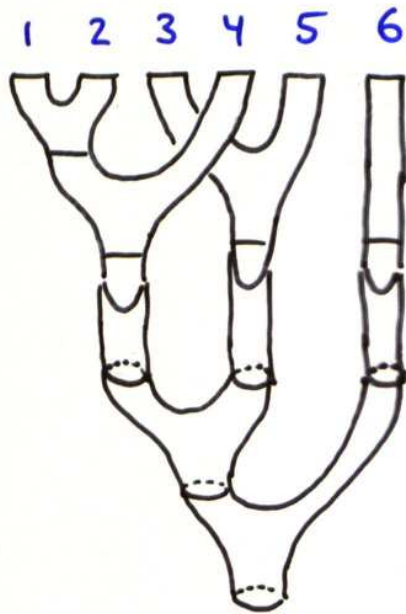
- (1) Find generators for the morphisms of 2Cob^{ext} (Morse thy)
- (2) relations are necessary, i.e. induced by $|\sim|$
- (3) relations are sufficient (normal form)
- (4) find invariants:
 - genus,
 - window number,
 - boundary permutation.

Normal form of an open-closed cobordism

$$M: \underbrace{(1, \dots, 1)}_m \rightarrow \underbrace{(0, \dots, 0)}_l$$

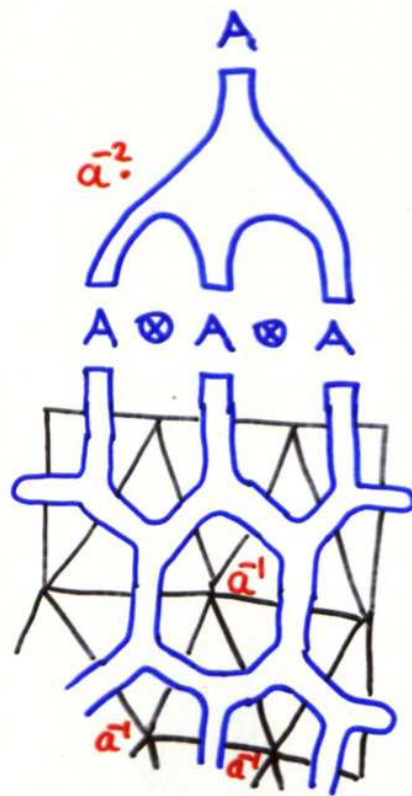
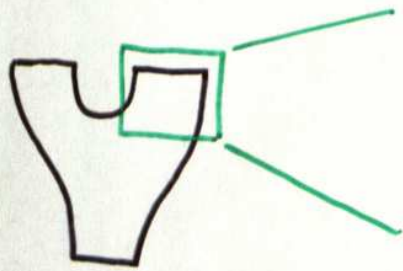


$m=6$



$$\sigma = (124)(35) \in S_6$$

boundary
permutation



$$P_3 = a^{-2} \cdot Z(\text{HH}) : A^{\otimes 3} \rightarrow A^{\otimes 3}$$

(idempotent)
 im $P_3 \cong A$

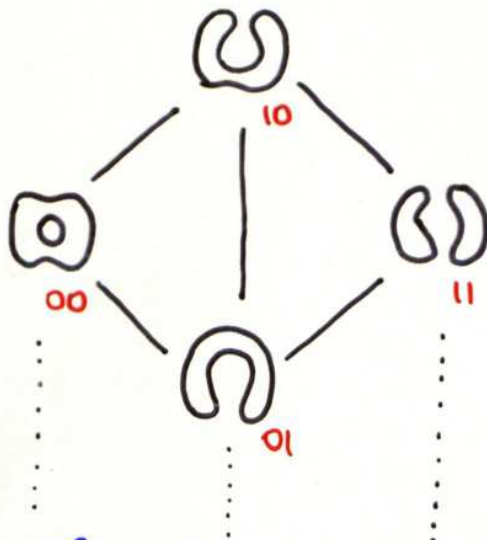
$$Q_1 = a^{-1} \cdot Z(\Phi) = a^{-1} \cdot Z(\text{figure}) : A \rightarrow A$$

(idempotent)
 im $Q_1 \cong Z(A)$

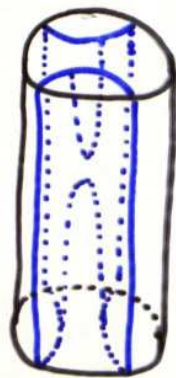
$$L = \text{link}(1, 1)$$

$$n_+ = 2$$

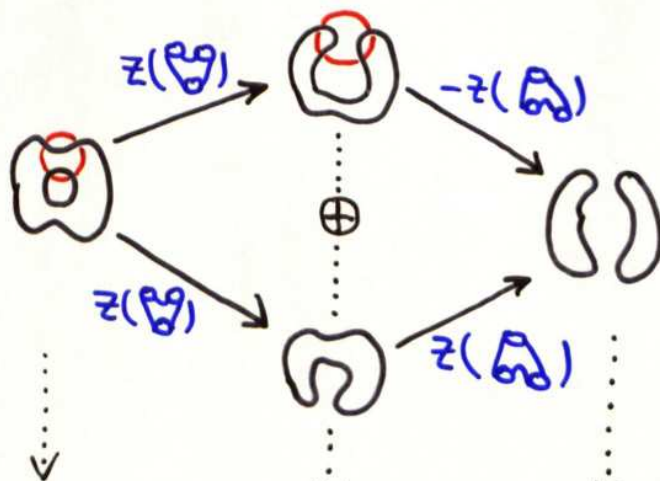
$$n_- = 0$$



$$\langle L \rangle = (q + q^{-1})^2 - q \cdot 2(q + q^{-1}) + q^2(q + q^{-1})^2$$



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$$C := z(s')$$

$$0 \rightarrow C \otimes C \rightarrow C \oplus C \rightarrow C \otimes C \rightarrow 0$$