2008-03-19

Invariants of 3d Kirby calculus and representation theory

Overview:

$$\begin{array}{c} \text{orientable connected closed 3-mfd} \\ \text{[Lickorish]} \uparrow \quad \downarrow \text{[Kirby]} \\ \text{surgery on a framed link in } S^3 / \text{Kirby relations} \end{array}$$

Q: Which invariants of framed links give invariants of 3-mfds?
(i.e. are also invariant under Kirby relations?)

example:
1) Coloured Jones polynomial, evaluated at even roots of unity, summed over all colours
2) Modular category:
finitely semisimple additive ribbon category $\mathcal{C}$ + non-degeneracy

Q: $\mathcal{C} \simeq R\text{-mod}$ for some ring R?

Thm: If $\mathcal{C}$ modular with $k = \mathrm{End}(\mathbb{1})$ a field, then

$$\mathcal{C} \simeq \text{comod-}H$$

H: finite-dimensional split cosemisimple weakly cofactorizable coribbon Weak Hopf Algebra over k with $H_t \cap H_s = k$.

today: explain the thm using example 1

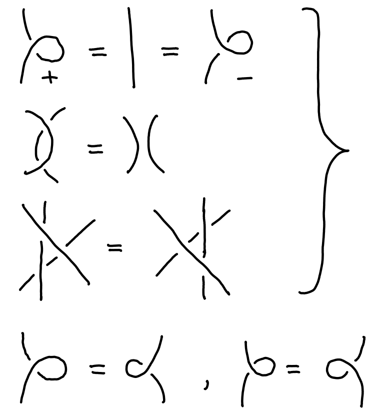
Reidemeister moves:

R_1

R_2

R_3

R_1'



ambient isotopy (R_1, R_2, R_3)

regular isotopy (R_2, R_3, R_1')

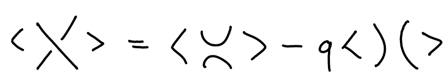
isotopy of framed links (diagrams in blackboard framing) (R_2, R_3, R_1')

Bracket polynomial:

L link, $\langle L \rangle \in \mathbb{Z}[q, q^{-1}]$

recursion:

$\langle O \rangle = q + q^{-1}$



satisfies R_2, R_3, R_1'

Kauffman bracket: $K(L) := \langle L \rangle|_{q=-A^{-2}} \cdot A^{n} \in \mathbb{Z}[A, A^{-1}]$ R_2, R_3, R_1'

Jones polynomial: $J(L) := (-1)^{n_- - n_+} q^{n_+ - 2n_-} \langle L \rangle / (q + q^{-1})|_{q=-\sqrt{t}} \in \mathbb{Z}[\sqrt{t}, \sqrt{t}^{-1}]$ R_1, R_2, R_3

(L oriented and non-empty)

n_+ n_-

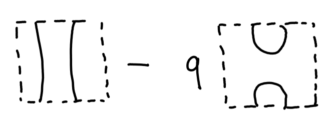
$n := n_+ + n_-$ crossing number

$w := n_+ - n_-$ writhe

$\langle$ diagram of framed link $\rangle \in \mathbb{Z}[q, q^{-1}]$

$\langle$ diagram of framed tangle $\rangle \rightarrow$ formal $\mathbb{Z}[q, q^{-1}]$-linear combination of crossing-free planar diagrams

e.g.



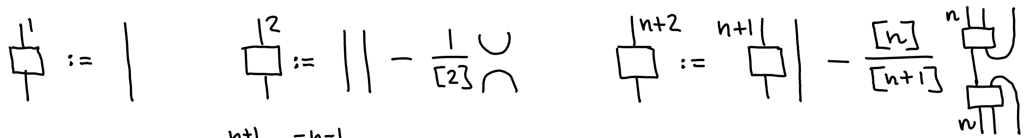
monoidal category: $\longrightarrow$ $\otimes$-product; $\downarrow$ "$\circ$" composition   (Temperley-Lieb category)

Now work over $\mathbb{Q}(q) \supseteq \mathbb{Z}[q, q^{-1}]$ (rational functions in q)

$\longrightarrow$ get $\mathbb{Q}(q)$-linear additive monoidal category

$|^n := \underbrace{||\cdots|}_{n}$ $\qquad |^0 :=$ nothing

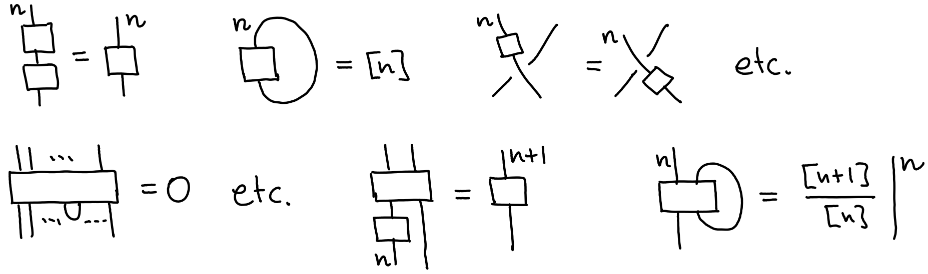
<u>Jones-Wenzl idempotents:</u>



$[n] := \dfrac{q^{n+1} - q^{-n-1}}{q - q^{-1}} \in \mathbb{Z}[q, q^{-1}]$ q-numbers , $[0] := 1$

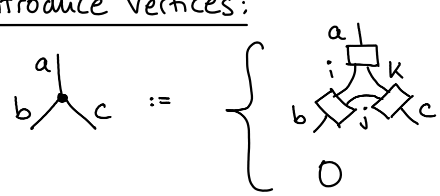
$q=1$ case: $(\mathbb{C}^2)^{\otimes n} \xrightarrow[\text{symmetrizer}]{} \mathbb{C}^{n+1}$ morphism of $U(\mathfrak{sl}_2)$-mod

properties:



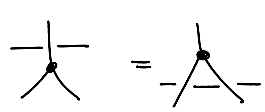
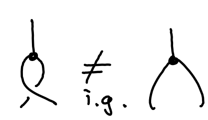
Extend $\langle L \rangle$ to links whose components are coloured by $0, 1, 2, \ldots$ by sticking  into each component $\longrightarrow$ coloured Kauffman bracket / Jones polynomial

<u>Introduce vertices:</u>



if $a+b-c,\ b+c-a,\ c+a-b \in \{0, 2, 4, 6, \ldots\}$ $\quad \begin{cases} i = \frac{a+b-c}{2} \\ j = \frac{b+c-a}{2} \\ k = \frac{c+a-b}{2} \end{cases}$

0 else

properties:  but: 

(vertex knows framing)

Roots of unity

$[1]=1 \quad [2]=q+q^{-1} \quad [3]=q^2+1+q^{-2}, \quad [4]=q^3+q+q^{-1}+q^{-3}, \ldots$

q root of unity → geometric sum → surprise !

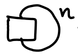
ε primitive $2p$-th root of unity

$\mathbb{Q}(\varepsilon)$ cyclotomic field, $\mathbb{Q}(q) \xrightarrow{q \mapsto \varepsilon} \mathbb{Q}(\varepsilon)$

↑ rational functions ↑ cyclotomic field

surprises:

1)  $= 0 \quad \forall\, n \geq p-1$ → restrict colours to $I = \{0,1,2,\ldots,p-2\}$

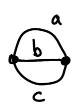
2) $\Delta_n :=$  $= [n] \neq 0$ if $n \in I$

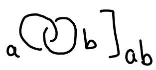
def: $(a,b,c) \in I^2$ admissible :⟺ (i) $a+b+c \equiv 0 \mod 2$

(ii) $a+b-c,\ b+c-a,\ c+a-b \geq 0$

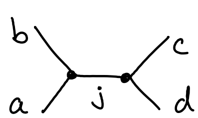
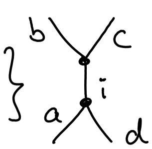
(iii) $a+b+c \leq 2p-4$

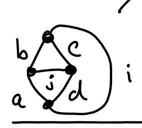
 $:= 0$ if (a,b,c) not admissible

3) $\Theta(a,b,c) :=$  $\neq 0$ if $(a,b,c) \in I^3$ admissible

4) $\left[\right.$$\left.\right]_{ab}$ is an invertible matrix in $\mathbb{Q}(\varepsilon)^{|I| \times |I|}$

Temperley-Lieb recoupling

 $= \sum_i \left\{ \begin{matrix} a & b & i \\ c & d & j \end{matrix} \right\}$ 

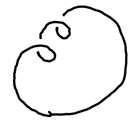
where $\left\{ \begin{matrix} a & b & i \\ c & d & j \end{matrix} \right\} =$  $\dfrac{\Delta_i}{\Theta(b,c,i)\,\Theta(a,d,i)}$

Sum over colours:

 $:= \sum_{j \in I} \Delta_j$ 

$\langle\langle L \rangle\rangle := \langle L$ with all components coloured $\omega \rangle$

3-Manifold invariant

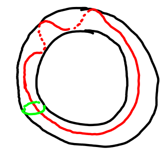


link component K: $S^1 \times \{0\} \overset{\varphi_K}{\hookrightarrow} S^3$

framed link component K: $S^1 \times D^2 \overset{\varphi_K}{\hookrightarrow} S^3$

$\partial(S^3 \setminus \varphi_K(S^1 \times D^2)) = S^1 \times S^1$

glue in $D^2 \times S^1$: $M_K := (S^3 \setminus \varphi_K(S^1 \times D^2)) \sqcup_{S^1 \times S^1} (D^2 \times S^1)$



$\varphi_K(S^1 \times D^2)$

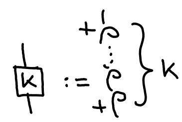
Thm: [Lickorish-Wallace]

Every connected orientable closed 3-mfd can be obtained from surgery on a framed link in S^3.

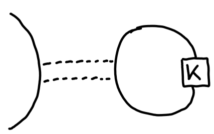
Thm: [Kirby]

Two framed links in S^3 represent the same 3-mfd iff they are related by a finite sequence of the following moves

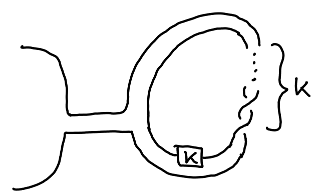
(i) isotopy of framed links

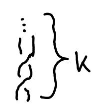


(ii)



$\longleftrightarrow$

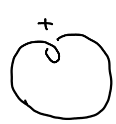




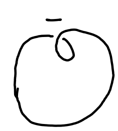
k = k double braidings

(2-handle slide)

(iii)



$\longleftrightarrow$ nothing $\longleftrightarrow$



($\# \mathbb{CP}^2$) ($\# \overline{\mathbb{CP}}^2$)

4d picture: $S^3 = \partial D^4$ D^4 = one 0-handle

$S^1 \times D^2 \overset{\varphi}{\hookrightarrow} S^3$ attaching map for 2-handle $D^2 \times D^2$

$\partial(D^2 \times D^2) \cap \partial M_K = D^2 \times S^1$

twisted longitude gets contractible over bdry of 2-handle $D^2 \times D^2$

Thm: [Reshetikhin-Turaev, Lickorish]

$Z(M_L) = \langle\langle L \rangle\rangle \, \langle\langle \circlearrowleft \rangle\rangle^{-b_+} \, \langle\langle \circlearrowright \rangle\rangle^{-b_-}$ is invariant under Kirby moves

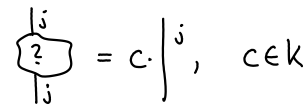
$b_\pm := \#$ pos/neg eigenvalues of $N_{j\ell} = \begin{cases} \ell k(K_j, K_\ell) & \text{if } j \neq \ell \\ w(K_j) & \text{if } j = \ell \end{cases}$ $[\tfrac{1}{2}(\times - \times)]$

Tannaka–Krein reconstruction

($\mathcal{C}$ = Temperley Lieb category, $k = \mathbb{Q}(\varepsilon)$, $\curlywedge$ admissible)

1) The objects $I = \{0, 1, \dots, p-2\}$ are simple, i.e. $\mathrm{End}(j) \cong k \quad \forall j \in I$

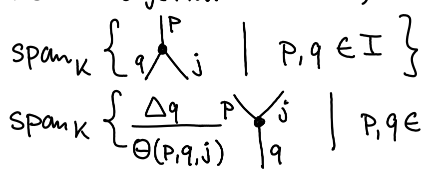
i.e.



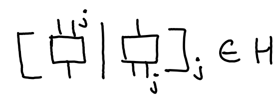
2) $F : \mathcal{C} \longrightarrow \mathrm{Vect}_k$ (canonical 'forgetful' functor)

$F(j) = \mathrm{Hom}(\Omega, \Omega \otimes j) = \mathrm{span}_k \{ \dots \mid p, q \in I \}$ $\qquad \Omega := \bigoplus_{j \in I} j$

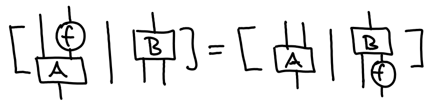
$F(j)^* = \mathrm{Hom}(\Omega \otimes j, \Omega) = \mathrm{span}_k \{ \dots \mid p, q \in I \}$



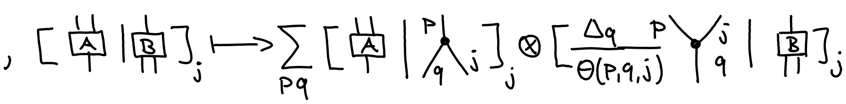
3) $H := \bigoplus_{j \in I} F(j)^* \otimes F(j) \quad (\mathrm{coend}(\mathcal{C}, F))$



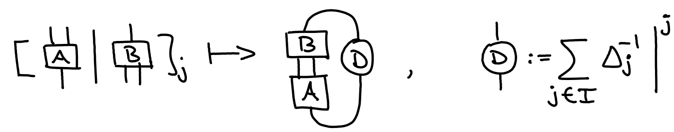
fact:



4) $\Delta : H \longrightarrow H \otimes H$,

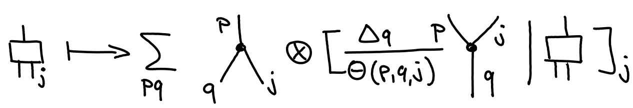


$\varepsilon : H \longrightarrow k$,



$\longrightarrow$ counital coassociative coalgebra

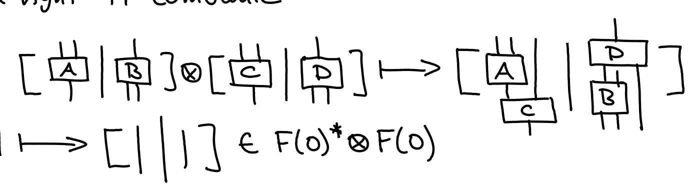
5) $\beta : F(j) \to F(j) \otimes H$,



$\longrightarrow$ $F(j)$ is a right-H-comodule

6) $\mu : H \otimes H \longrightarrow H$,

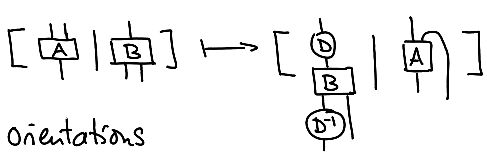
$\eta : k \longrightarrow H$,



$\longrightarrow$ unital associative algebra, controls the $\otimes$-product of $\mathcal{C}$

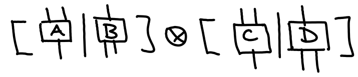
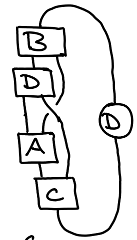
$$\beta_{j \otimes \ell} : F(j) \otimes F(\ell) \longrightarrow F(j) \otimes F(\ell) \otimes H$$
$$v \otimes w \longmapsto (\mathrm{id} \otimes \mu) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\beta_j \otimes \beta_\ell)(v \otimes w)$$
$$F(j \otimes \ell) := F(j) \hat{\otimes} F(\ell) = \ker(\mathrm{id} - (\mathrm{id} \otimes (\varepsilon \circ \mu)) \circ \beta_{j \otimes \ell}) \leq F(j) \otimes F(\ell)$$

7) $S : H \longrightarrow H$,

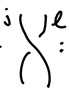


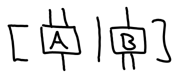
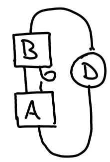
$\longrightarrow$ controls orientations

8) $(H, \mu, \eta, \Delta, \varepsilon, S)$ finite-dimensional split cosemisimple Weak Hopf Algebra

9) $r : H \otimes H \longrightarrow k$,  $\longmapsto$ 

$\rightarrow$ coquasitriangular structure, controls crossings

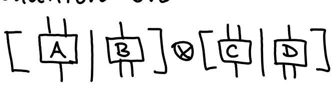
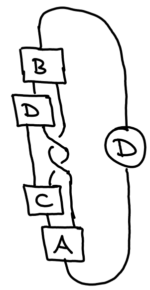
$(\tau \otimes r) \circ \beta_{j \otimes \ell} =$  $: F(j \otimes \ell) \longrightarrow F(\ell \otimes j)$

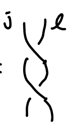
10) $\nu : H \longrightarrow k$,  $\longmapsto$ 

$\rightarrow$ universal ribbon twist, controls framing

$(id \otimes \nu) \circ \beta_j =$  $: F(j) \longrightarrow F(j)$

11) Non-degeneracy condition on

$q : H \otimes H \longrightarrow k$,  $\longmapsto$  (weak cofactorizability)

$(id \otimes q) \circ \beta_{j \otimes \ell} =$  $: F(j \otimes \ell) \longrightarrow F(j \otimes \ell)$

$\rightarrow$ invertibility of