Tangle homology from 2d extended TQFTs

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Open-closed cobordisms

Open-closed TQFTs

Knowledgeable Frobenius algebras

Examples

Tangle homology

Tangle composition

Strong separability

TQFTs in Khovanov homology

$\hat{J}(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L \rangle$

$\langle \ \rangle = 1$

$\langle O \rangle = q + q^{-1}$

$\langle \times \rangle = \langle)(\rangle - q \langle \asymp \rangle$

"categorify"

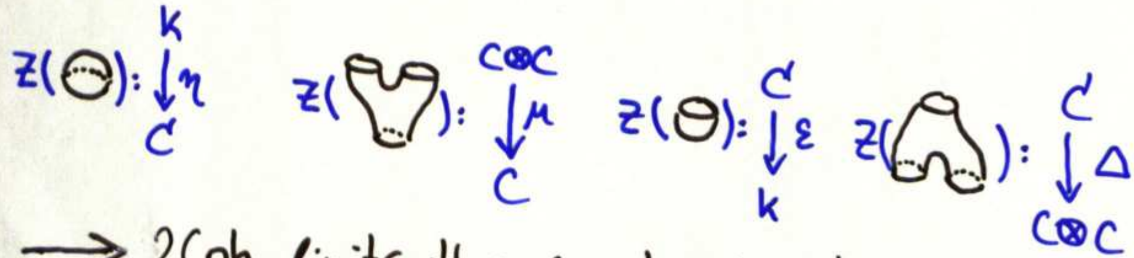
$\cdots \longrightarrow C^j \longrightarrow C^{j+1} \longrightarrow \cdots$

def: A 2-dimensional TQFT is a symmetric monoidal functor

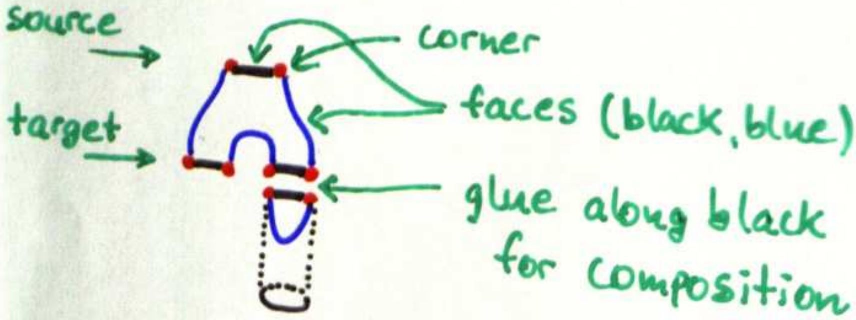
$Z: 2\mathrm{Cob} \longrightarrow \mathrm{Vect}_k$

thm: [Abrams, Sawin]
2Cob is the free symmetric monoidal category generated by a commutative Frobenius algebra object.

$C := Z(O)$



$\longrightarrow$ 2Cob limits the construction to links idea: use $2\mathrm{Cob}^{ext}$ "open-closed" cobordisms



$2\mathrm{Cob}^{ext}$ consists of

objects: compact 1-manifolds with boundary

morphisms: generated by

subject to the Moore-Segal relations

<u>def:</u> An open-closed TQFT is a symmetric monoidal functor

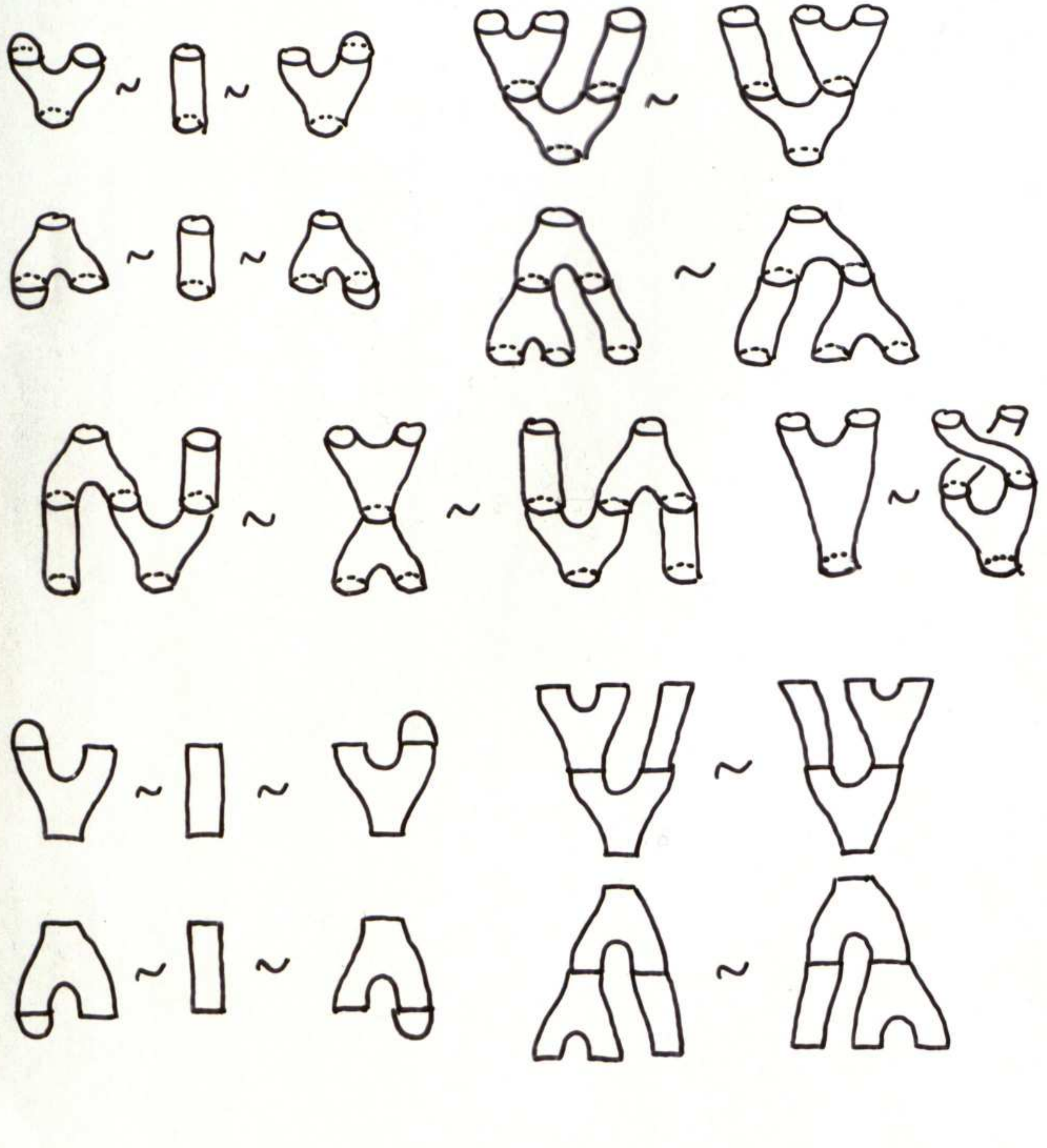
$$Z: 2\mathrm{Cob}^{ext} \longrightarrow \mathrm{Vect}_k$$

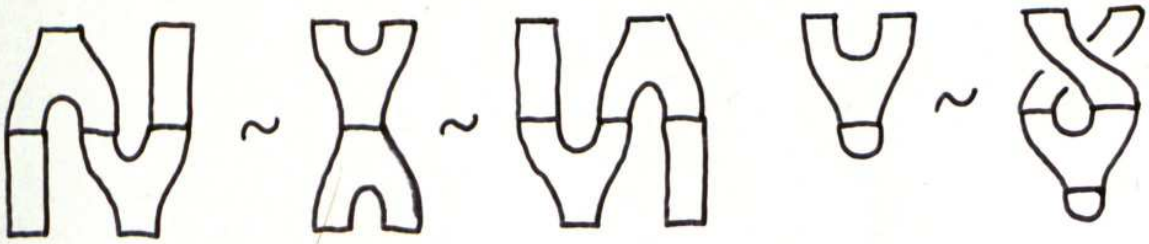
<u>thm:</u> [AL, HP]

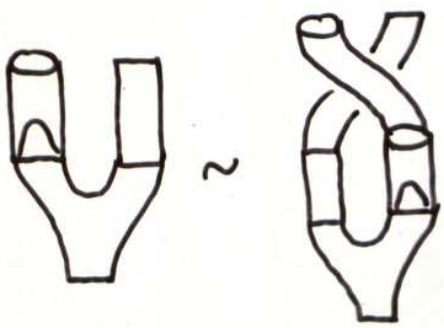
$2\mathrm{Cob}^{ext}$ is the free symmetric monoidal category generated by a knowledgeable Frobenius algebra object.

η_C μ_C Δ_C ε_C η_A μ_A Δ_A ε_A ι^* ι

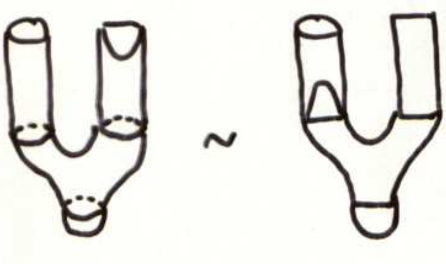
The following morphisms of $2\mathrm{Cob}^{\mathrm{ext}}$ are equivalent:



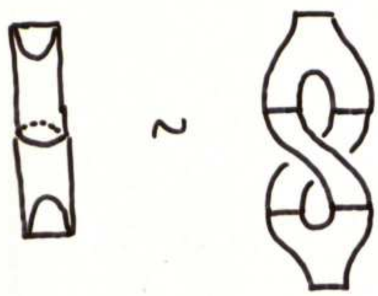




Knowledge



duality



Cardy condition

def: A knowledgeable Frobenius algebra (A, C, ι, ι^*) consists of

(1) a symmetric Frobenius algebra $(A, \mu_A, \eta_A, \Delta_A, \varepsilon_A)$
(2) a commutative Frobenius algebra $(C, \mu_C, \eta_C, \Delta_C, \varepsilon_C)$
(3) linear maps $\iota: C \to A$ and $\iota^*: A \to C$

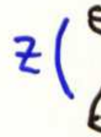
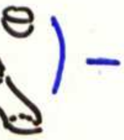
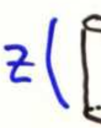
such that

(a) $\iota: C \to A$ is a homomorphism of algebras
(b) $\mu_A \circ (\iota \otimes id_A) = \mu_A \circ \tau_{A,A} \circ (\iota \otimes id_A)$ [Knowledge]
(c) $\varepsilon_C \circ \mu_C \circ (id_C \otimes \iota^*) = \varepsilon_A \circ \mu_A \circ (\iota \otimes id_A)$ [duality]
(d) $\mu_A \circ \tau_{A,A} \circ \Delta_A = \iota \circ \iota^*$ [Cardy condition]

→ Find examples for which C is such that

$Z($  $) = 0$ [S]

$Z($  $) = 2$ [T]

$Z($  $) + Z($  $) - Z($  $) - Z($  $) = 0$ [4Tu]

<u>example</u>: (A, C, ι, ι^*) with char $k = p$ and

$$C = k[x]/(x^2) \qquad \varepsilon_C(1) = 0 \qquad \Delta_C(1) = 1 \otimes x + x \otimes 1$$
$$\varepsilon_C(x) = 1 \qquad \Delta_C(x) = x \otimes x$$

(Khovanov's example)

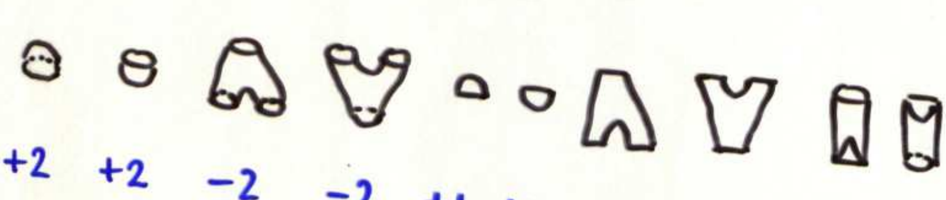
$$A = k[y]/(y^p) \qquad \varepsilon_A(y^\ell) = 0 \;\; \forall \ell < p-1 \qquad \Delta_A(y^\ell) = \sum_{j=0}^{p-1-\ell} y^{\ell+j} \otimes y^{p-1-j}$$
$$\varepsilon_A(y^{p-1}) = 1$$

(truncated polynomial algebra)

$$\iota(1) = 1 \qquad \iota^*(y^\ell) = 0 \;\; \forall \ell < p-1$$
$$\iota(x) = 0 \qquad \iota^*(y^{p-1}) = 1$$

<u>def</u>: Grading and filtration extend to $2\mathrm{Cob}^{\mathrm{ext}}$ with

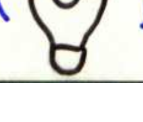
M



deg: $+2$ $+2$ -2 -2 $+1$ $+1$ -1 -1 -1 -1

$$\deg(M) = 2\chi(M) - |\pi_0(\text{blue face})| + \#\text{windows}$$

<u>prop</u>: If (A, C, ι, ι^*) is graded | filtered and C is Khovanov's | Lee's | Bar-Natan's Frobenius algebra, then

$Z(\Theta) = 0$ and $Z($  $) = 0$

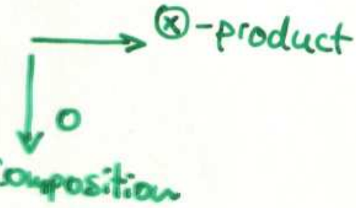
$[\![\uparrow]\!] = 0 \to A \to 0$

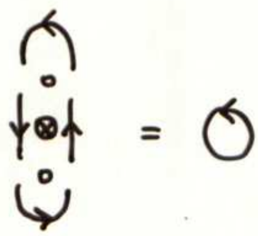
$[\![\circlearrowleft]\!] = 0 \to C \to 0$

[[]] $= 0 \to A\otimes A\{2\} \xrightarrow{Z(\,\cdot\,)} A \otimes A\{4\} \to 0$

Operations on tangle diagrams

$\downarrow \qquad \uparrow \qquad \cup \qquad \cap$



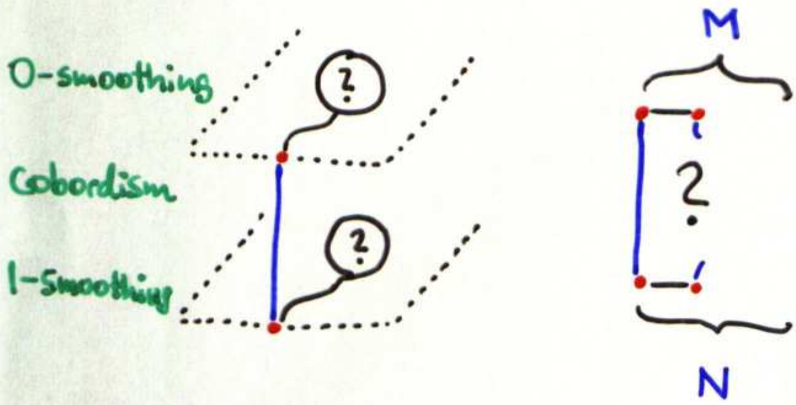


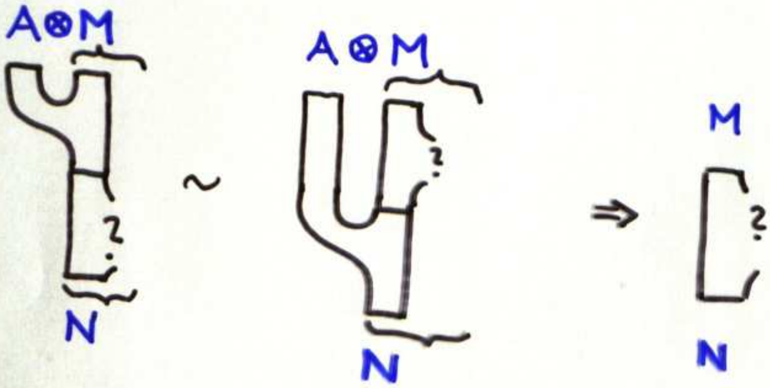
$$[\![\cup]\!] \circ \left([\![\downarrow]\!] \otimes [\![\uparrow]\!]\right) \circ [\![\cap]\!] \overset{?}{=} [\![\circlearrowleft]\!]$$

What are these
Operations on complexes?

"$\otimes$" is $\otimes_k$ of complexes from symmetric monoidal structure of the TQFT

composition of tangles





is a morphism of left-A -modules

$\rightarrow$ one action of A for each boundary point of the tangle

$\longleftrightarrow$ complex of $(A^{\otimes p}, A^{\otimes q})$-bimodules

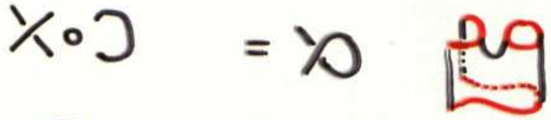
$$[\![T_{rq} \circ T_{qp}]\!] = [\![T_{rq}]\!] \otimes_{A^{\otimes q}} [\![T_{qp}]\!]$$

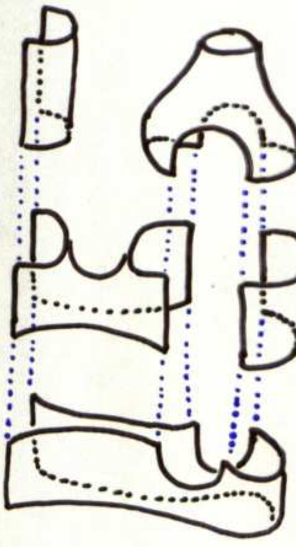
consider the coequalizer

$$M \otimes A \otimes N \underset{id_M \otimes \lambda_N}{\overset{\rho_M \otimes id_N}{\rightrightarrows}} M \otimes N \xrightarrow{\otimes_A} M \otimes_A N$$

$\rho_M : M \otimes A \to M$
$\lambda_N : A \otimes N \to N$

→ make sure it is the correct complex
→ especially not just zero!

example: 



} factors through this one

} ⊗ of complexes

} coequalizer map $\otimes_A$

<u>thm:</u> A sufficient condition for composition to work is that

$Z($  $) : k \to A$

is a unit of A ($\Leftrightarrow$ A is strongly separable) example: $h, t \in k$

$$C_{h,t} = k[x]/(x^2 - hx - t)$$

$$\Delta(1) = 1 \otimes x + x \otimes 1 - h 1 \otimes 1 \qquad \varepsilon(1) = 0$$
$$\Delta(x) = x \otimes x + t 1 \otimes 1 \qquad \varepsilon(x) = 1$$

is strongly separable $\Leftrightarrow$ $h^2 + 4t \neq 0$

$C_{0,0}$	(Khovanov's)	not strongly separable
$C_{0,1}$	(Lee's)	strongly separable iff char $k \neq 2$
$C_{1,0}$	(Bar-Natan's)	strongly separable

prop: If $C_{h,t}$ is not strongly separable and not a field, then there is no (A, C, ι, ι^*) with A strongly separable.

Summary: Tangle composition works algebraically for extensions of $C_{h,t}$ with $h^2 + 4t \neq 0$, but not for $C_{0,0}$.

WHY ?

__input:__ Complex of filtered modules with $C_{0,1}$ (Lee's)

__output:__ spectral sequence with

E_0	Khovanov's chain complex	
E_1	Khovanov's homology	Reidemeister move invariant
E_2	secondary homology	
$\vdots$		
E_n	Lee's homology	

$\longrightarrow$ Taking this spectral sequence is not compatible with the composition of tangle diagrams !