Fusion categories in terms of graphs and relations

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Quantum 3-manifold invariants

Monoidal categories with extra structure:

- Turaev–Viro: Finitely semisimple spherical category $\mathcal{C}$
- Reshetikhin–Turaev: Modular category $\mathcal{C}$

Two sorts of tensor products:

- If $\mathcal{C} \simeq H-\text{Mod}$, H Hopf algebra: easy tensor product
- Otherwise: difficult tensor product

Observation:

- easy tensor product $\longleftrightarrow$ homotopy type invariant
- difficult tensor product $\longleftrightarrow$ stronger invariant

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This talk

- is about the difficult $\otimes$-products
- applies to all (multi-) fusion categories
- gives a combinatorial description of the $\otimes$-product
- example: modular categories associated with $U_q(\mathfrak{sl}_2)$

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Recall $SL_q(2)$

$SL_q(2) = \mathbb{C}\{\, t_{ij} \mid 1 \leq i, j \leq 2 \,\}/(\text{relations})$

- quadratic relations $RTT - TTR$

$$\sum_{k,\ell} \left(R^{k\ell}_{ij} t^p_k t^q_\ell - t^k_i t^\ell_j R^{pq}_{k\ell} \right), \quad R = \begin{pmatrix} q & & & \\ & 0 & 1 & \\ & 1 & q - q^{-1} & \\ & & & q \end{pmatrix}$$

- inhomogeneous relation (degree 0 and 2)

$$1 - \text{qdet}, \qquad \text{qdet} = da - qbc$$

- $\implies \mathcal{M}^{SL_q(2)} \simeq {}_{U_q(\mathfrak{sl}_2)}\mathcal{M}$

We will get an analogous description of (multi-)fusion categories.

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Fusion categories

Definition

A multi-fusion category $\mathcal{C}$

- is monoidal $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \rho, \lambda)$
- is k-linear (k field), additive (all finite biproducts)
- has $\mathrm{Hom}(X, Y)$ finite-dimensional over k for all $X, Y \in |\mathcal{C}|$
- is autonomous (each object is equipped with a left-dual)
- is finitely semisimple
- has $\mathrm{End}(X) \cong k$ for all simple $X \in |\mathcal{C}|$
- $\implies$ is abelian

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Canonical forgetful functor

Theorem (Hayashi, Hai)

$\mathcal{C}$ (multi-)fusion category.

- Small projective generator $\hat{V} = \bigoplus_{j \in J} V_j$
- The *long canonical forgetful functor*

$$\omega \colon \mathcal{C} \to \mathbf{Vect}_k, \qquad X \mapsto \mathrm{Hom}(\hat{V}, \hat{V} \otimes X)$$

is k-linear, faithful and exact.

Theorem (classical)

$H = \mathbf{coend}(\mathcal{C}, \omega) = \bigoplus_{j \in J} (\omega V_j)^* \otimes \omega V_j$

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Weak Hopf Algebra structure

Theorem (Szlachányi, HP)

$\mathcal{C}$ (multi-)fusion category. Then $\omega\colon \mathcal{C} \to \mathbf{Vect}_k$

- is lax and oplax monoidal
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Theorem (Hayashi, Hai, Ostrik, HP)

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The Dimension Graph

Definition

$\mathcal{C}$ (multi-)fusion, M monoidal generator of $\mathcal{C}$.

The finite directed *dimension graph* $\mathcal{G}$ of $\mathcal{C}$:

- vertices: J (classes of simple objects)
- edges $j \to \ell$: basis of $\mathrm{Hom}(V_j, V_\ell \otimes M)$

Notation

- $\mathcal{G}^0$: vertices
- $\mathcal{G}^m$: paths of length $m \geq 1$

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Modular categories associated with $U_q(\mathfrak{sl}_2)$

Example

- $p = 3, 4, 5, \ldots$
- $\zeta^{4p} = 1$ (primitive $4p$-th root of unity)
- Field $k = \mathbb{C}$ or $k = \mathbb{Q}(\zeta)$
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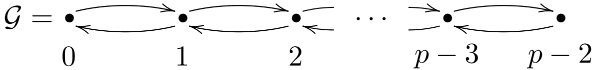

The Path Algebra of $\mathcal{G} \times \mathcal{G}$

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$$H[\mathcal{G}] = k(\mathcal{G} \times \mathcal{G}) = \bigoplus_{m \geq 0} \underbrace{k\mathcal{G}^m \otimes k\mathcal{G}^m}_{H[\mathcal{G}]_m}$$

- is a graded associative unital algebra over k
- $p, q \in \mathcal{G}^m$: $[p|q]_m = p \otimes q \in H[\mathcal{G}]_m$ (basis elements)
- is a direct sum of matrix coalgebras
- is a split cosemisimple coassociative counital coalgebra

Theorem

$H[\mathcal{G}]$ is a graded, split cosemisimple Weak Bialgebra.

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$H[\mathcal{G}]$ Rules all Fusion Categories

Theorem

There is a surjection of Weak Bialgebras

$$\pi\colon H[\mathcal{G}] \to \mathbf{coend}(\mathcal{C},\omega)$$

such that

- $\pi((k\mathcal{G}^0)^* \otimes k\mathcal{G}^0) = (\omega\mathbb{1})^* \otimes \omega\mathbb{1}$
- $\pi((k\mathcal{G}^1)^* \otimes k\mathcal{G}^1) = (\omega M)^* \otimes \omega M$

and for the comodules

- $k\mathcal{G}^m$ pushes forward to $(\omega M)^{\otimes m}$

Next Step

Compute the kernel $I = \ker\pi$ and obtain a characterization

$$H[\mathcal{G}]/I \cong \mathbf{coend}(\mathcal{C},\omega) \quad \text{and} \quad \mathcal{M}^{H[\mathcal{G}]/I} \simeq \mathcal{C}$$

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Modular categories associated with $U_q(\mathfrak{sl}_2)$

Example

I is the two-sided ideal generated by:

- quadratic relations $RTT - TTR$:

$$\sum_{(def)\in\mathcal{G}^2}\left(R_{(abc),(def)}[def|pqr]_2 - [abc|def]_2 R_{(def),(pqr)}\right)$$

- $\longrightarrow$ guarantee that $k\mathcal{G}^m$ decomposes as $(\omega M)^{\otimes m}$
- inhomogeneous relation (degree 0 and degree 2):

$$1 - \mathrm{qdet}$$

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Example (... continued)

Non-zero coefficients of $R_{(abc),(def)}$:

$$
\begin{aligned}
R_{(j,j\pm1,j);(j,j\pm1,j)} &= \mp\zeta^{-1}\,\frac{q^{\pm(j+1)}}{[j+1]_q}, \\
R_{(j,j-1,j);(j,j+1,j)} &= \zeta^{-1}\,\frac{[j]_q[j+2]_q}{[j+1]_q^{\,2}}, \\
R_{(j,j+1,j);(j,j-1,j)} &= \zeta^{-1}, \\
R_{(j,j\pm1,j\pm2);(j,j\pm1,j\pm2)} &= \zeta^{-1}q^{-1},
\end{aligned}
$$

with $q=\zeta^2$, $[n]_q=(q^n-q^{-n})/(q-q^{-1})$.

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Example (... continued)

Quantum determinant:

$$\mathrm{qdet} = \sum_{j,\ell=0}^{r-2} \Bigg(\frac{[\ell+1]_q}{[j+1]_q}\,[(j,j+1,j)|(\ell,\ell+1,\ell)]_2 + \frac{[\ell]_q}{[j]_q}\,[(j,j-1,j)|(\ell,\ell-1,\ell)]_2 - \frac{[\ell+1]_q}{[j]_q}\,[(j,j-1,j)|(\ell,\ell+1,\ell)]_2 - \frac{[\ell]_q}{[j+1]_q}\,[(j,j+1,j)|(\ell,\ell-1,\ell)]_2 \Bigg).$$

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The quotient $H = H[\mathcal{G}]/I$

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Summary

Theorem

Every multi-fusion category $\mathcal{C}$ is of the form $\mathcal{C} \simeq \mathcal{M}^{H[\mathcal{G}]/I}$ where $\mathcal{G}$ is its dimension graph with respect to some fusion generator $M \in |\mathcal{C}|$ and the two-sided ideal I is generated by two types of relations

- type 1 (homogeneous): implement $\mathrm{End}((\omega M)^{\otimes m})$ for all $m = 0, 1, 2, \ldots$
- type 2 (inhomogeneous): remove suitable group-likes and make $H[\mathcal{G}]/I$ a finite-dimensional Weak Hopf Algebra

You can make the tensor product of every fusion category trivial provided that you understand the ideal I above.

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