2d extended TQFTs and categorification

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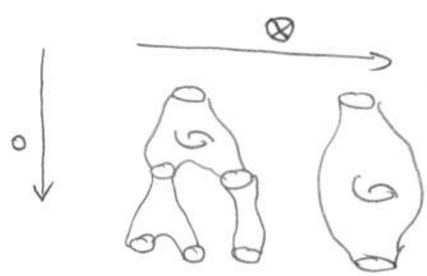
1. 2d TQFTs
2. mfds with corners and 2d extended TQFTs
   → canonical form, invariants, classification
3. Khovanov homology for tangles

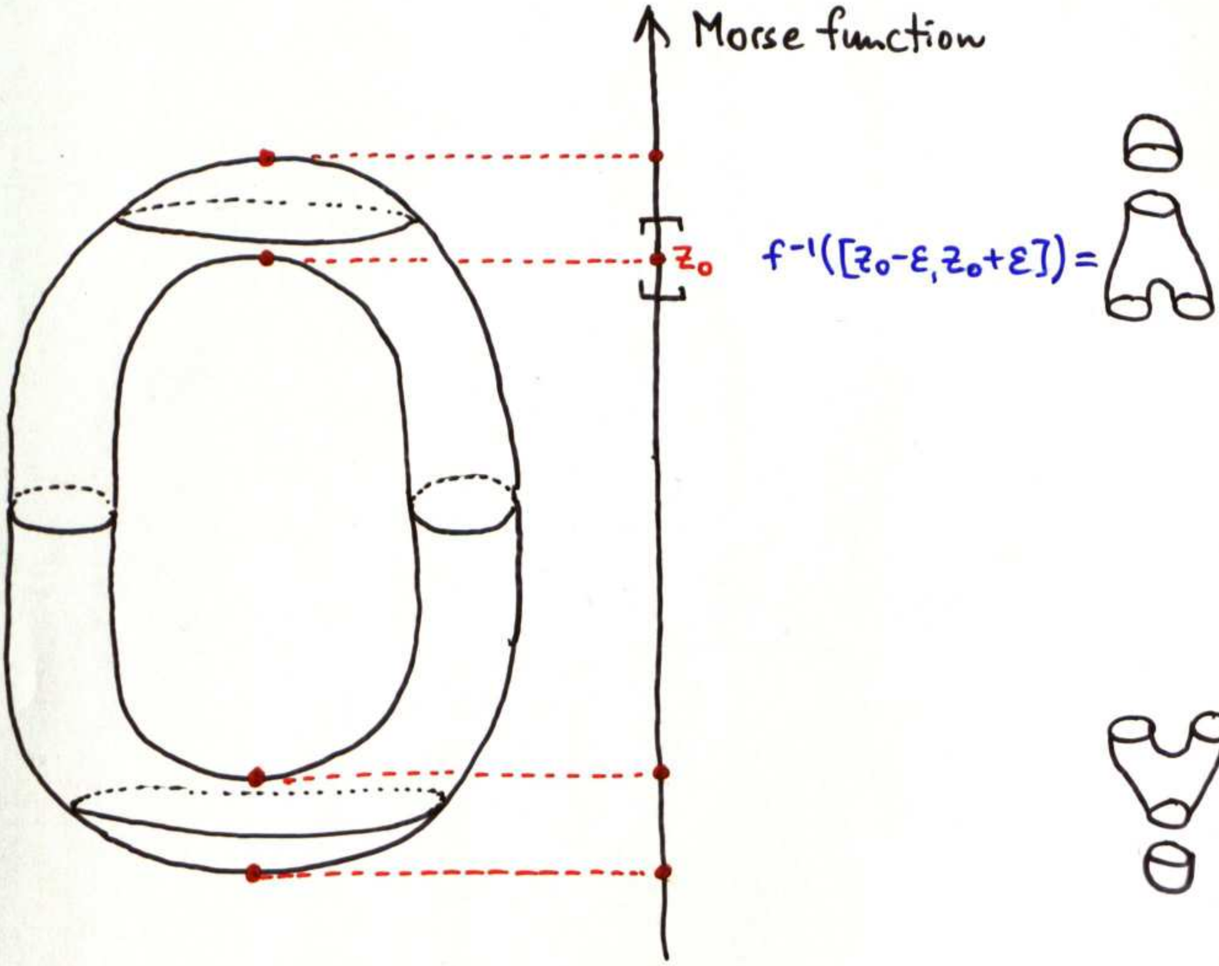
<u>def:</u> A d-dimensional TQFT is a symmetric monoidal functor

$$Z: d\mathrm{Cob} \to \mathrm{Vect}_K.$$

Here: $d=2$.



$T^2 \subseteq \mathbb{R}^3$



A 2-dimensional TQFT sends

(1) closed 1-manifolds to k-vector spaces

 $\longmapsto$ C

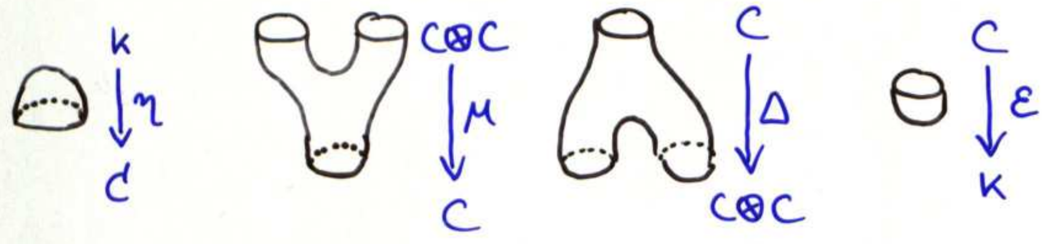
 $\longmapsto$ $C \otimes C$

$\emptyset \longmapsto k$

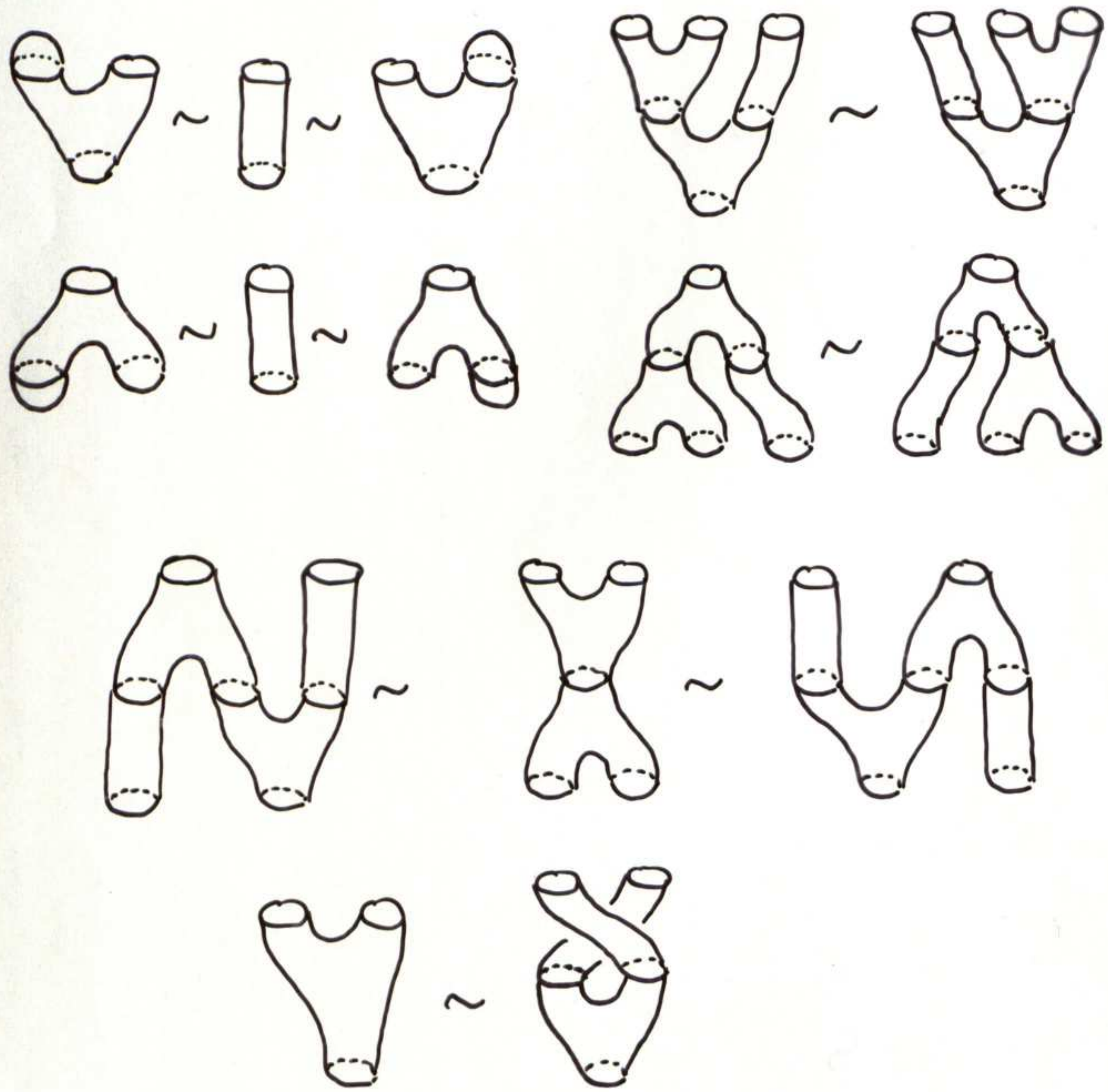
(2) 2-dimensional cobordisms to k-linear maps

 $\longmapsto$ $k \xrightarrow{\eta} C$

 $\longmapsto$ $C \otimes C \xrightarrow{\mu} C$



The following morphisms of 2Cob are equivalent:



thm: [Dijkgraaf, Segal, Abrams, Sawin]

2Cob is the free strict symmetric monoidal category generated by a Commutative Frobenius algebra object.

$\Rightarrow$ 2d TQFTs are classified by commutative Frobenius algebras:

$$\underbrace{\mathrm{SymMon}[2\mathrm{Cob}, \mathrm{Vect}_k]}_{2d\,TQFTs} \simeq \mathrm{comFrob}_k$$

def: A Frobenius algebra $(C, \mu, \eta, \Delta, \varepsilon)$ is a vector space C with linear maps

$$\mu: C\otimes C \to C, \quad \eta: k \to C, \quad \Delta: C \to C\otimes C, \quad \varepsilon: C \to k$$

such that

1) (C, μ, η) is a unital associative algebra
2) (C, Δ, ε) is a counital coassociative coalgebra

i.e. $(id\otimes\varepsilon)\circ\Delta = id = (\varepsilon\otimes id)\circ\Delta$

$(\Delta\otimes id)\circ\Delta = (id\otimes\Delta)\circ\Delta$

3) $(id\otimes\mu)\circ(\Delta\otimes id) = \Delta\circ\mu = (\mu\otimes id)\circ(id\otimes\Delta)$

It is called commutative if

$\mu\circ\tau = \mu$ $\qquad \tau(a\otimes b) = b\otimes a$

It is called symmetric if

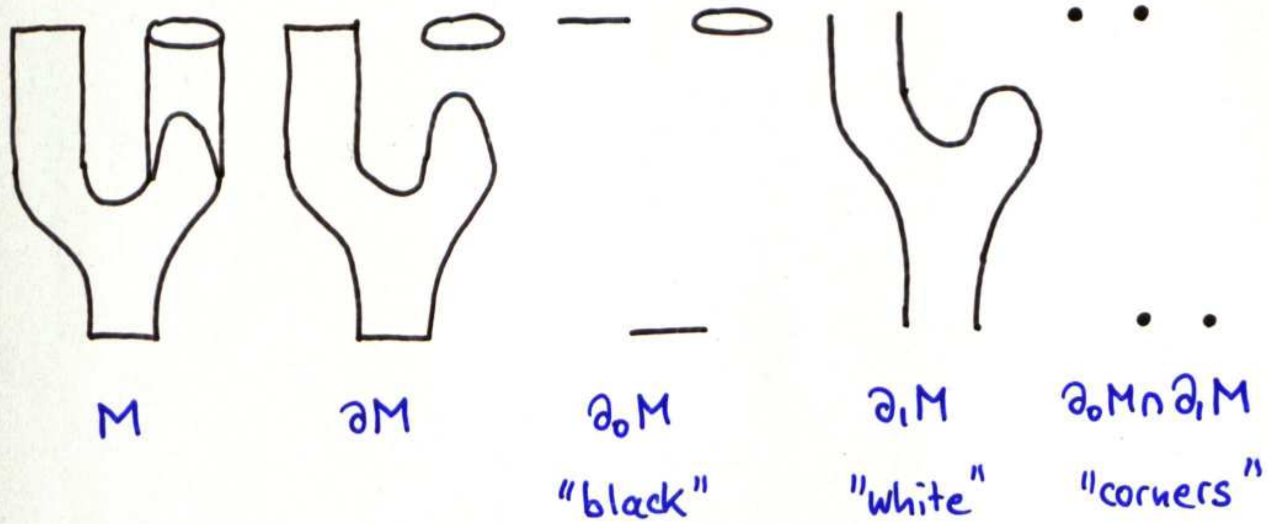
$\varepsilon\circ\mu\circ\tau = \varepsilon\circ\mu.$

ex: $C = k[x]/(x^2)$

$\Delta(1) = 1\otimes x + x\otimes 1 \qquad \varepsilon(1) = 0$

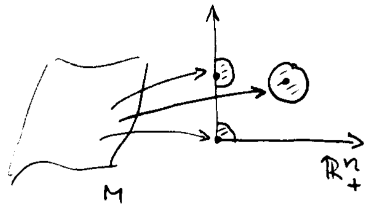
$\Delta(x) = x\otimes x \qquad \varepsilon(x) = 1$

A 2-dimensional <2>-manifold:



remark: A smooth manifold with corners M has coordinate systems $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ such that the charts are homeomorphisms

$$\varphi_\alpha : U_\alpha \longrightarrow \varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n_+ := [0,\infty)^n$$



and transition functions

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

are restrictions to $\mathbb{R}^n_+$ of diffeomorphisms.

def: $p \in U_\alpha \subseteq M$

$$c(p) = \#\{ i \mid (\varphi_\alpha(p))_i = 0 \}$$ (number of zero coefficients)

A connected face is the closure of a component of $\{p \in M \mid c(p) = 1\}$

A face is a union of pairwise disjoint connected faces.

A manifold with faces is a smooth manifold with corners such that each $p \in M$ is contained in $c(p)$ different connected faces

A <2>-manifold $(M, \partial_0 M, \partial_1 M)$ is a manifold with faces M with faces $\partial_0 M, \partial_1 M$ such that

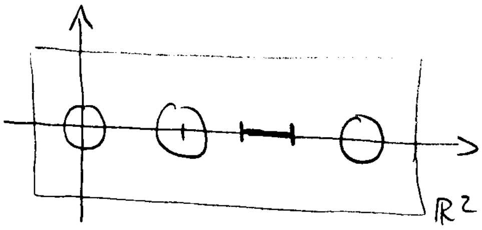
(i) $\partial M = \partial_0 M \cup \partial_1 M$

(ii) $\partial_0 M \cap \partial_1 M$ is a face of both $\partial_0 M$ and $\partial_1 M$.

$\ell \in \mathbb{N}_0$

$\underline{n} = (n_0, \dots, n_{\ell-1}) \in \{0,1\}^\ell$

$$\Sigma_{\underline{n}} := \dot{\bigcup_{j=0}^{\ell-1}} \Sigma^{(n_j)}((j,0)) \subseteq \mathbb{R}^2 \qquad \begin{array}{l} \Sigma^{(0)}(x,y) := \partial B_{1/4}(x,y) \\ \Sigma^{(1)}(x,y) := [x-1/4, x+1/4] \times \{y\} \end{array}$$

$\Sigma_{0,0,1,0} =$ 

<u>def:</u> The category $2\mathrm{Cob}^{ext}$ of 2-dimensional <u>open-closed cobordisms</u> consists of

objects: $\underline{n} \in \{0,1\}^\ell$, $\ell \in \mathbb{N}_0$

morphisms: $[(M, \psi_1, \psi_2)] : \underline{n} \to \underline{n}'$

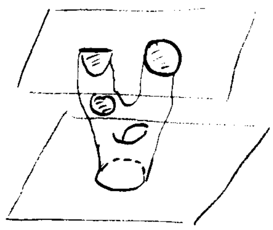
M compact oriented 2-dimensional $\langle 2 \rangle$-manifold

$$\left.\begin{array}{l} \psi_1 : \Sigma^*_{\underline{n}} \to \psi_1(\Sigma^*_{\underline{n}}) \subseteq \partial_0 M \\ \psi_2 : \Sigma_{\underline{n}'} \to \psi_2(\Sigma_{\underline{n}'}) \subseteq \partial_0 M \end{array}\right\} \text{orientation-preserving diffeo}$$

such that

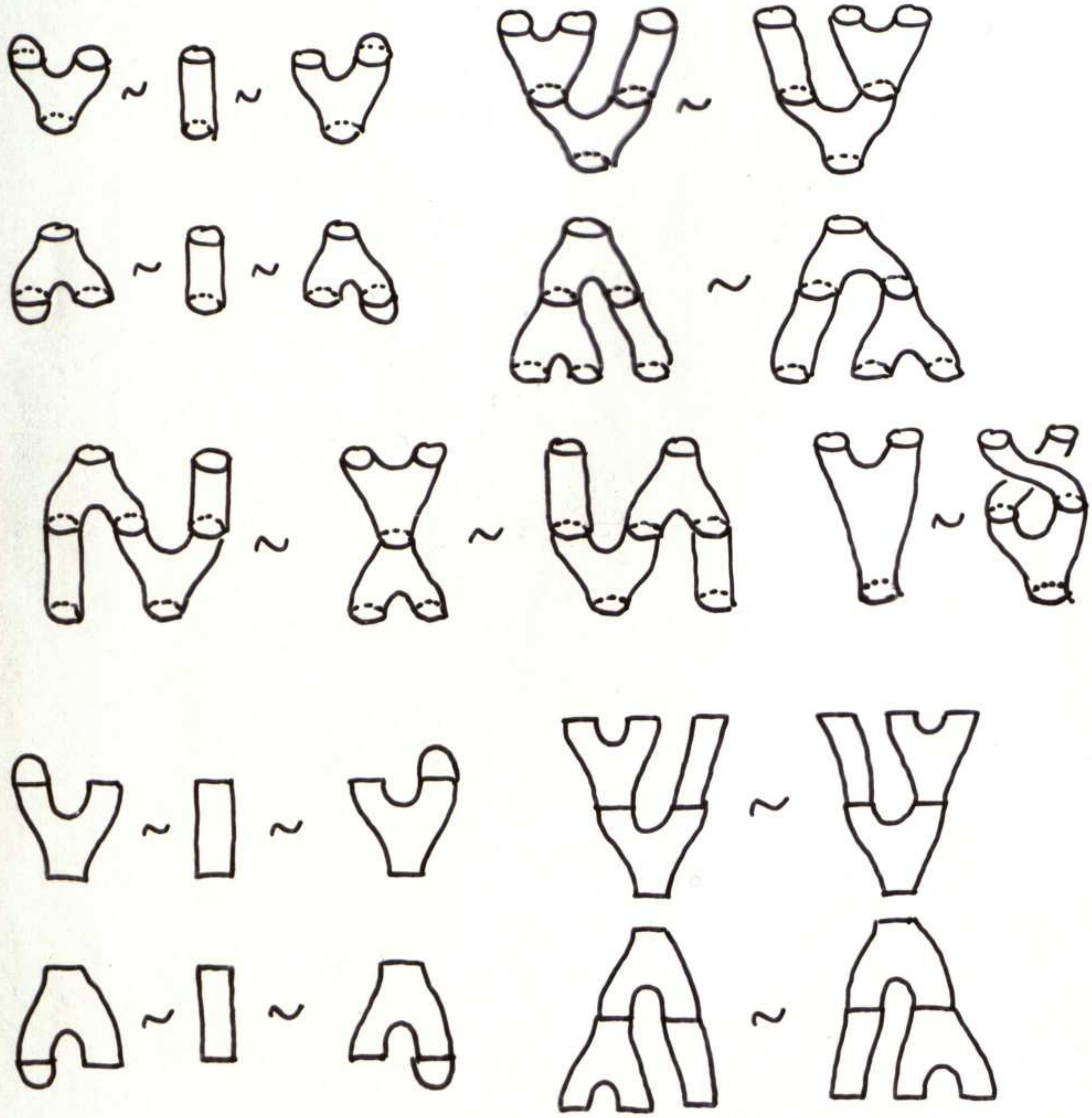
$$\partial_0 M = \psi_1(\Sigma^*_{\underline{n}}) \,\dot{\cup}\, \psi_2(\Sigma_{\underline{n}'})$$

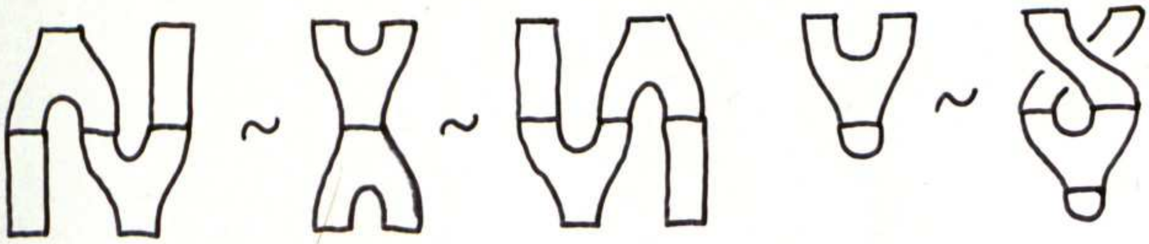
"$\sim$" as above.

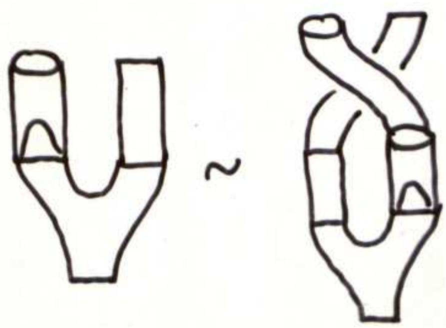
<u>picture:</u> 

<u>def:</u> A 2-dimensional <u>open-closed TQFT</u> is a symmetric monoidal functor $Z : 2\mathrm{Cob}^{ext} \to \mathrm{Vect}_K$.

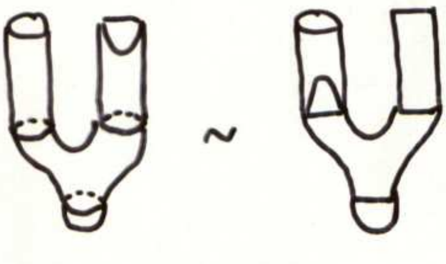
The following morphisms of $2\mathrm{Cob}^{\mathrm{ext}}$ are equivalent:



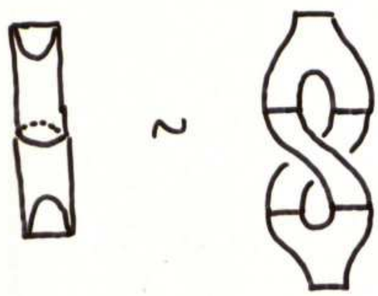




Knowledge



duality



Cardy condition

def: A knowledgeable Frobenius algebra (A, C, ι, ι^*) consists of

a) a symmetric Frobenius algebra $(A, \mu_A, \eta_A, \Delta_A, \varepsilon_A)$

b) a commutative Frobenius algebra $(C, \mu_C, \eta_C, \Delta_C, \varepsilon_C)$

c) linear maps $\iota : C \to A$, $\iota^* : A \to C$

such that

1) ι is a homomorphism of algebras

2) $\mu_A \circ (\iota \otimes id_A) = \mu_A \circ \tau \circ (\iota \otimes id_A)$

3) $\varepsilon_C \circ \mu_C \circ (id_C \otimes \iota^*) = \varepsilon_A \circ \mu_A \circ (\iota \otimes id_A)$

4) $\iota \circ \iota^* = \mu_A \circ \tau \circ \Delta_A$

thm: The category $2Cob^{ext}$ is equivalent to the free strict symmetric monoidal category generated by a knowledgeable Frobenius algebra.

$\Rightarrow$ $$\mathrm{SymMon}[2Cob^{ext}, \mathrm{Vect}_K] \simeq K\text{-}\mathrm{Frob}_K$$

by the way: operad? No.

PROP? No.

multicategory? Yes.

Symmetric monoidal sketch? Yes.

proof: 1) Find generators for the morphisms of $2Cob^{ext}$ (Morse thy)

2) relations are necessary, i.e. induced by '$\sim$'

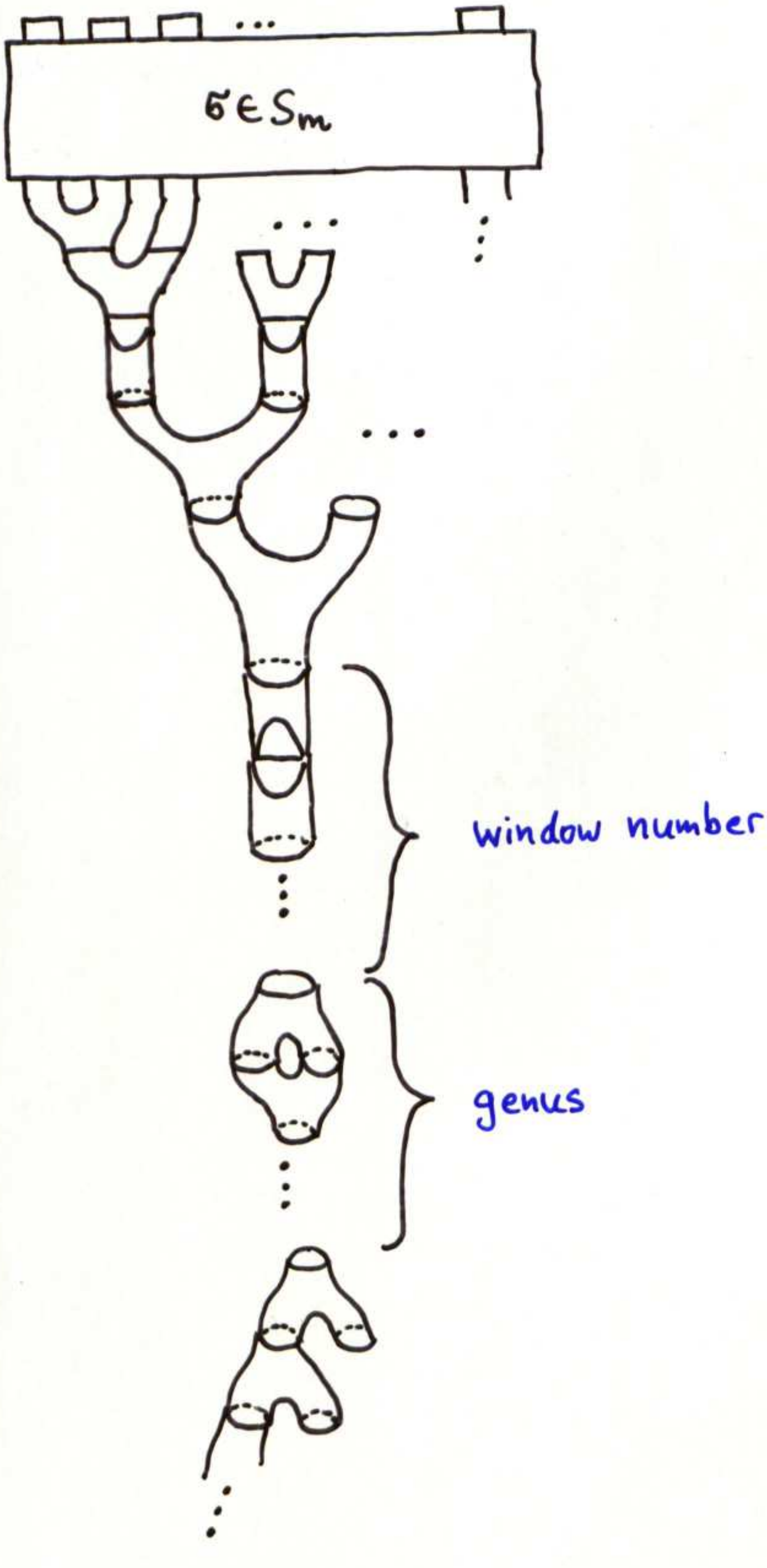
3) relations are sufficient (canonical form)

4) Complete set of invariants:

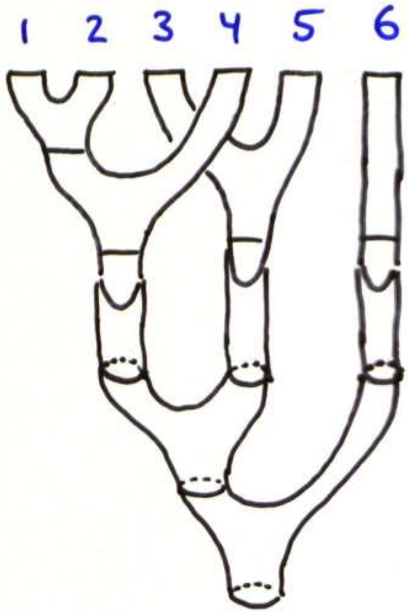
genus, window number, boundary permutation, # bdry components

Normal form of an open-closed cobordism

$$M: \underbrace{(1,\dots,1)}_{m} \to \underbrace{(0,\dots,0)}_{\ell}$$



$m=6$



$\sigma = (124)(35) \in S_6$

boundary permutation

Khovanov homology

L plane projection of oriented link

n_+ n_- $n = n_+ + n_-$ # crossings

Kauffman bracket: $\langle L \rangle \in \mathbb{Z}[q, q^{-1}]$

$$\langle \bigcirc \rangle = q + q^{-1}$$

$$\langle \times \rangle = \underset{0-}{\langle \asymp \rangle} - q \underset{1-\text{smoothing}}{\langle)(\rangle}$$

unnormalized Jones polynomial

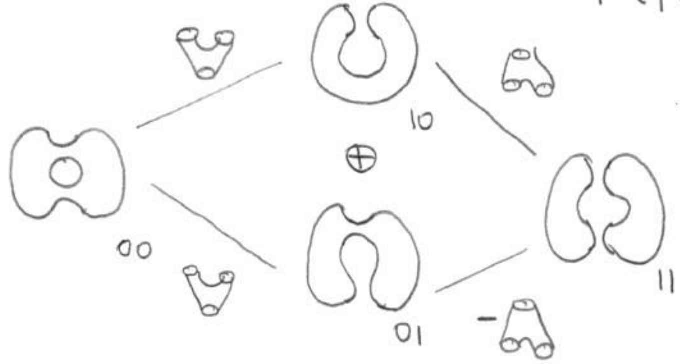
$$\hat{J}(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L \rangle$$

$\hat{J}(L) = ($

$\langle L \rangle = (q+q^{-1})^2 - q\ 2(q+q^{-1}) + q^2(q+q^{-1})^2\ ) \cdot q^2 = q^4 + q^2 + 1 + q^{-2}$

$L =$ 

$n_+ = 2$
$n_- = 0$



$$[\![L]\!] = \Big(0 \longrightarrow C \otimes C \longrightarrow C \oplus C_{\{1\}} \longrightarrow C \oplus C_{\{2\}} \longrightarrow 0\Big)\underbrace{\{2\}}_{\text{shift in } q\text{-degree}}$$

$C = K[x]/(x^2)$

$\operatorname{qdim} C = \underset{x}{q} + \underset{1}{q^{-1}}$, $\chi([\![L]\!]) = \sum_j (-1)^j \operatorname{qdim} [\![L]\!]_j$

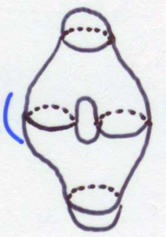
Reidemeister moves

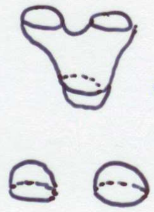
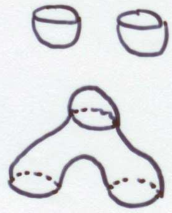
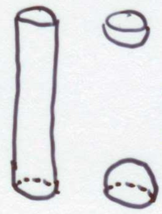
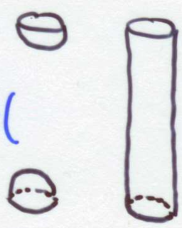
$$\begin{array}{ccc} L_1 & \hat{J}(L_1) & [\![L_1]\!] \\ \Big\updownarrow \text{move} & \| \text{ equal} & \wr\!\wr \text{ htpy equiv} \\ L_2 & \hat{J}(L_2) & [\![L_2]\!] \end{array}$$

thm: a) Given a K-Frob (A, C, ι, ι^*) such that C satisfies S, T, 4Tu, one gets for each plane diagram of an oriented (p,q)-tangle T a complex $[\![T]\!]$ of $(A^{\otimes p}, A^{\otimes q})$-bimodules whose htpy type is an invariant.

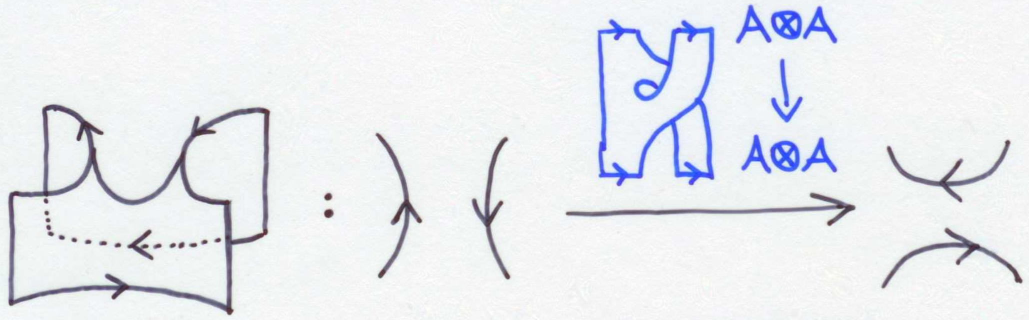
b) If A is strongly separable, tangle composition is $\otimes_{A^{\otimes p}}$ of complexes of bimodules, and there is a semistrict braided monoidal 2-category with weak duals whose decategorification gives the invariant of (a).

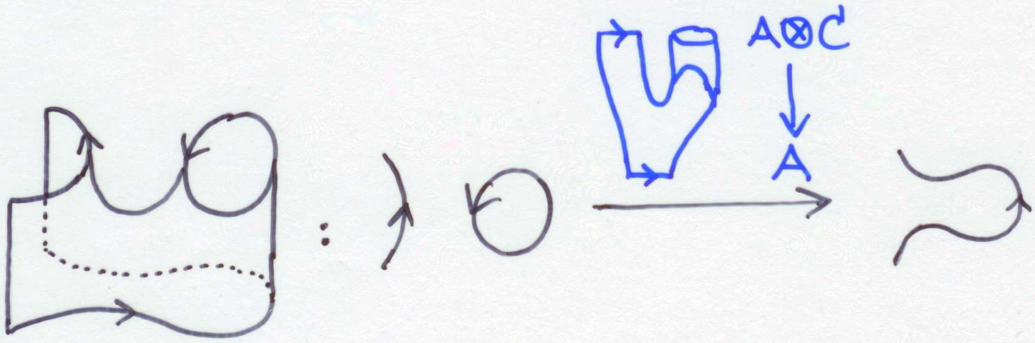
Z() = 0 [S]

Z() = 2 [T]

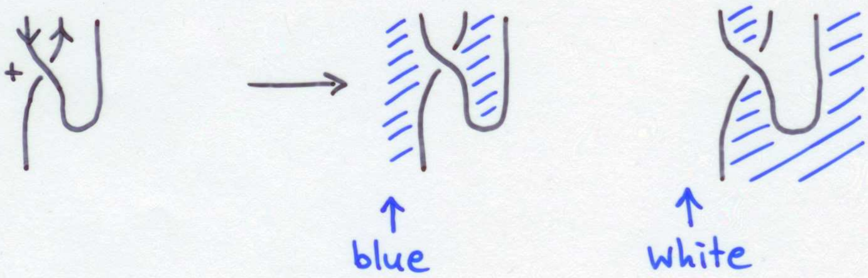
Z() + Z() − Z() − Z() = 0 [4Tu]

→ turn Bar-Natan's Cobordisms into algebra





Two checkerboard colourings



Orient the smoothings:

"counter-clockwise around blue"