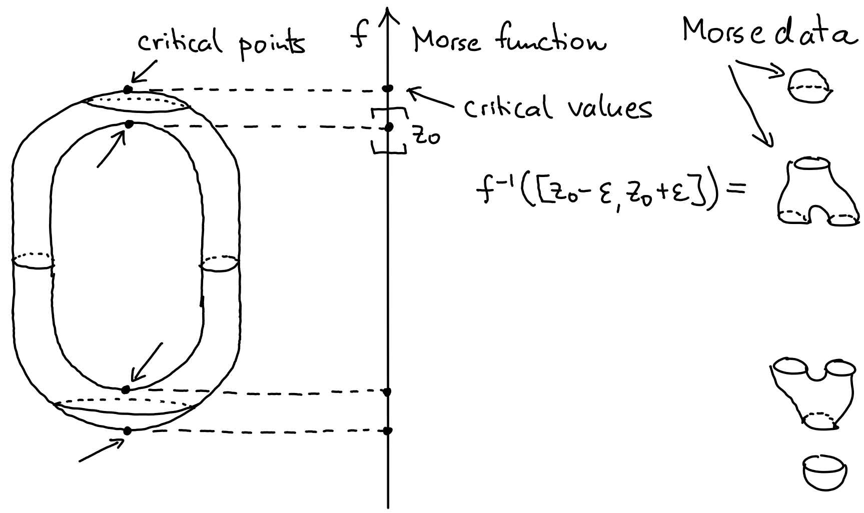
2d TQFTs and extended TQFTs

1 Introduction

Morse theory: $T^2 \subseteq \mathbb{R}^3$



rough idea: A 2-dimensional TQFT sends

1) closed 1-manifolds to k-vector spaces

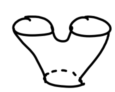
 $\longmapsto$ C

 $\longmapsto$ $C \otimes C$

$\emptyset \longmapsto k$

2) 2-dimensional cobordisms to k-linear maps

 $\longmapsto$ $k \xrightarrow{\eta} C$

 $\longmapsto$ $C \otimes C \xrightarrow{\mu} C$

→ classify cobordisms

→ use vector spaces and linear maps to compute invariants

→ translate from topology to algebra

Outline:

2 2d 'closed' TQFTs

3 Manifolds with corners

4 2d open-closed TQFTs

5 big picture

2 2d closed TQFTs

<u>Def:</u> The category $n\mathrm{Cob}$ of n-dimensional cobordisms consists of

objects: closed oriented smooth $(n-1)$-manifolds

morphisms: $\Sigma \xrightarrow{M} \Sigma'$ compact oriented smooth n-manifolds M with $\partial M = \Sigma^* \amalg \Sigma'$ up to orientation preserving diffeomorphism rel ∂.

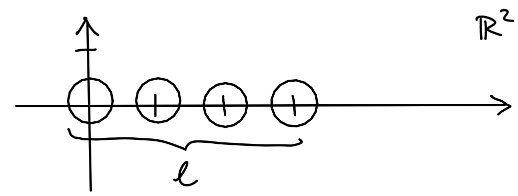
<u>Def:</u> An n-dimensional TQFT is a symmetric monoidal functor

$$Z: n\mathrm{Cob} \to \mathrm{Vect}_K$$

$\to$ show $n=2$ in detail.

$\ell \in \mathbb{N}_0$, $\Sigma_\ell := \coprod_{j=0}^{\ell-1} \partial B_{1/4}((j,0)) \subseteq \mathbb{R}^2$ (induced orientation)

Σ_ℓ^* (opposite orientation)



<u>Prop:</u> (A skeleton of) the category $2\mathrm{Cob}$ of 2-dimensional cobordisms consists of

objects: $\ell \in \mathbb{N}_0$

morphisms: $[(M, \psi_1, \psi_2)] : \ell \to \ell'$

M compact oriented smooth 2-manifold

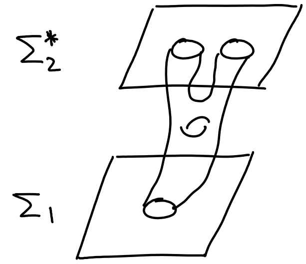
$\left.\begin{array}{l} \psi_1 : \Sigma_\ell^* \to \psi_1(\Sigma_\ell^*) \subseteq \partial M \\ \psi_2 : \Sigma_{\ell'} \to \psi_2(\Sigma_{\ell'}) \subseteq \partial M \end{array}\right\}$ orientation preserving diffeomorphisms

such that $\partial M = \psi_1(\Sigma_\ell^*) \amalg \psi_2(\Sigma_{\ell'})$

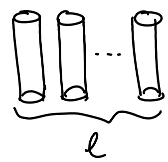
$(M, \psi_1, \psi_2) \sim (N, \varphi_1, \varphi_2)$ iff $\exists\, f: M \to N$ orientation preserving diffeomorphism such that

$$\begin{tikzcd} & M \arrow[dd, "f"] & \\ \Sigma_\ell^* \arrow[ur, "\psi_1"] \arrow[dr, "\varphi_1"'] & & \Sigma_{\ell'} \arrow[ul, "\psi_2"'] \arrow[dl, "\varphi_2"] \\ & N & \end{tikzcd}$$

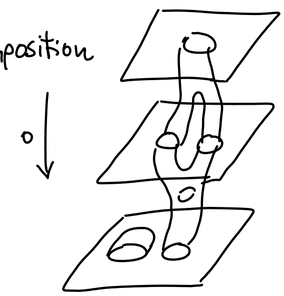
picture:



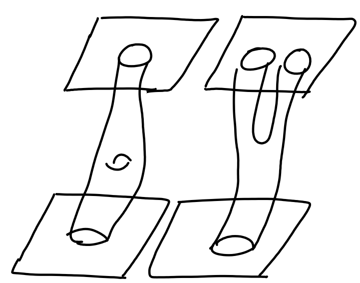
as abstract manifolds

$id_{\Sigma_\ell} =$ 

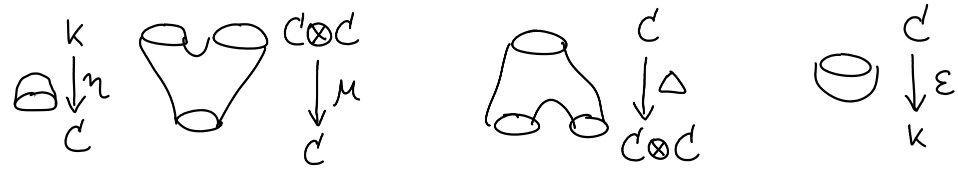
composition



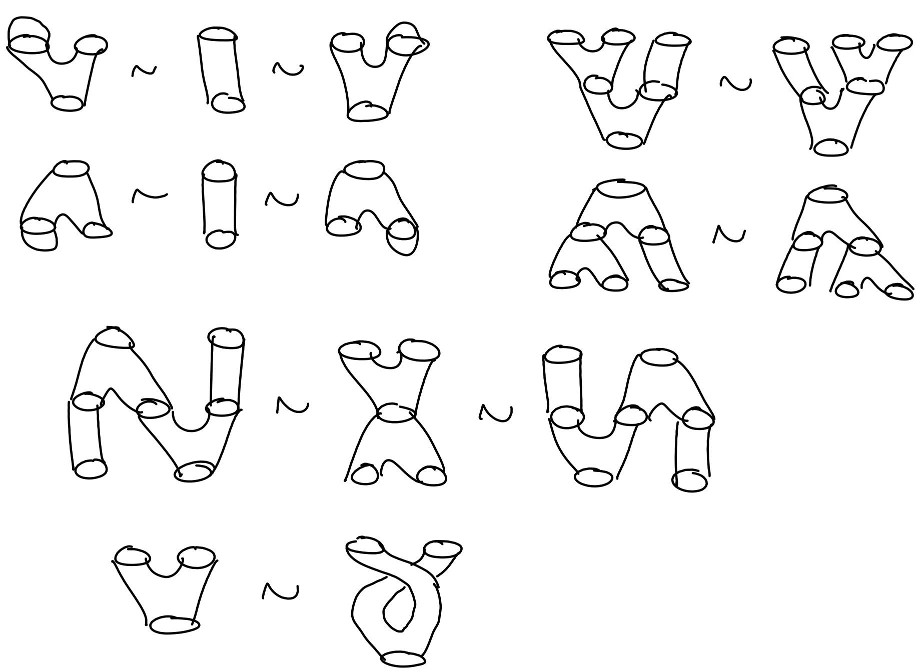
$\otimes$ tensor product



Morse data:



Equivalent morphisms of 2Cob:



Def: A <u>Frobenius algebra</u> $(C,\mu,\eta,\Delta,\varepsilon)$ is a k-vector space C with linear maps

$$\eta: k \to C, \quad \mu: C\otimes C \longrightarrow C, \quad \Delta: C \to C\otimes C, \quad \varepsilon: C \to k$$

such that

(1) $\quad (C,\mu,\eta)$ unital associative algebra

(2) $\quad (C,\Delta,\varepsilon)$ counital coassociative coalgebra

(i) $(id\otimes\varepsilon)\circ\Delta = id = (\varepsilon\otimes id)\circ\Delta$

(ii) $(id\otimes\Delta)\circ\Delta = (\Delta\otimes id)\circ\Delta$

(3) $\quad (id\otimes\mu)\circ(\Delta\otimes id) = \Delta\circ\mu = (\mu\otimes id)\circ(id\otimes\Delta)$

It is called <u>commutative</u> if

$$\mu\circ\tau = \mu \qquad\qquad \tau(a\otimes b) = b\otimes c$$

It is called <u>symmetric</u> if

$$\varepsilon\circ\mu\circ\tau = \varepsilon\circ\mu.$$

Thm: [Dijkgraaf, Abrams, Sawin]

2Cob is the free symmetric monoidal category generated by a commutative Frobenius algebra object.

$$\Rightarrow \qquad \text{2d-TQFT} := \text{SymMon}[\text{2Cob}, \text{Vect}_k] \simeq \text{ComFrob}$$

More details on the choice of a skeleton:

<u>Def:</u> A category $\mathcal{C}$ is <u>skeletal</u> if any two isomorphic objects are equal.
A <u>skeleton</u> of a category $\mathcal{D}$ is a skeletal full subcategory.

<u>Fact:</u> Every category is equivalent to every of its skeleta.

<u>Prop:</u> 1) For each orientation preserving diffeomorphism $f: \Sigma_0 \to \Sigma_1$, there is a cobordism $M_f = \Sigma_0 \times I : \Sigma_0 \longrightarrow \Sigma_1$

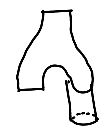
2) Two such diffeomorphisms $f, f': \Sigma_0 \longrightarrow \Sigma_1$ yield equivalent cobordisms $M_f \sim M_{f'}$ $\iff$ $f \simeq f'$ smoothly homotopic.

3) Given a TQFT $Z: n\mathrm{Cob} \longrightarrow \mathrm{Vect}_K$ and an object $\Sigma \in |n\mathrm{Cob}|$,

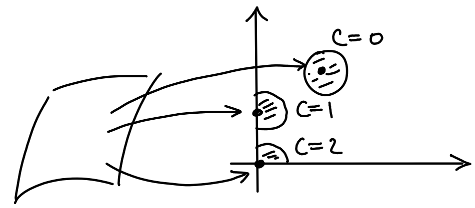
$$\mathrm{Diff}\,\Sigma / (\mathrm{Diff}_0 \Sigma) \circlearrowright Z(\Sigma).$$

4) Two closed oriented smooth 1-manifolds Σ_0, Σ_1 are diffeomorphic $\iff$ $\exists$ invertible cobordism $M: \Sigma_0 \longrightarrow \Sigma_1$.

3 Manifolds with corners

aim: Extend 2d cobordisms to manifolds like this: 

Remark: A smooth manifold with corners M has coordinate systems $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ such that the charts are homeomorphisms



$$\varphi_\alpha : U_\alpha \longrightarrow \varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n_+ := [0,\infty)^n$$

and transition functions

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta)$$

are restrictions to $\mathbb{R}^n_+$ of diffeomorphisms.

Def: $p \in U_\alpha \subseteq M$

$c(p) = |\{i \mid (\varphi_\alpha(p))_i = 0\}|$ (number of zero coefficients)

A connected face is the closure of a component of $\{p \in M \mid c(p) = 1\}$.

A face is a union of pairwise disjoint connected faces.

A manifold with faces is a smooth manifold with corners such that each $p \in M$ is contained in $c(p)$ different connected faces.

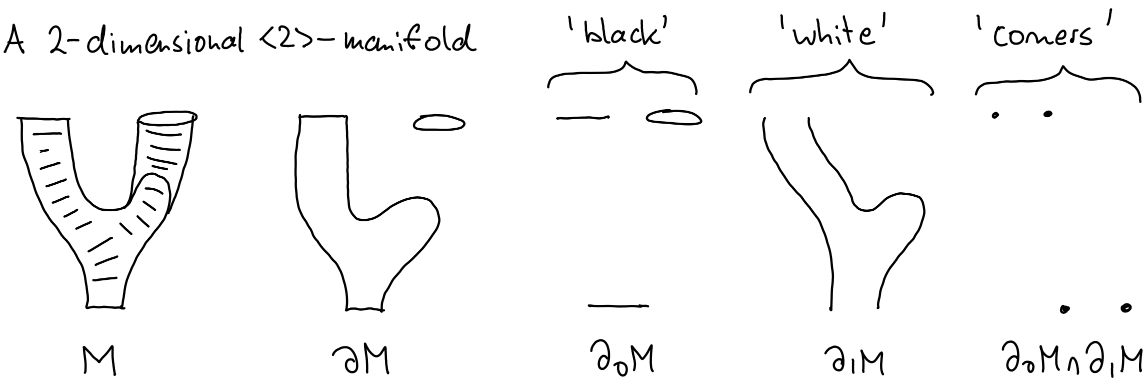
$\longrightarrow$ rules out 

A $\langle 2 \rangle$-manifold $(M, \partial_0 M, \partial_1 M)$ is a manifold with faces M with faces $\partial_0 M, \partial_1 M$ such that

(a) $\partial M = \partial_0 M \cup \partial_1 M$

(b) $\partial_0 M \cap \partial_1 M$ is a face of both $\partial_0 M$ and $\partial_1 M$.

Example: A 2-dimensional $\langle 2 \rangle$-manifold

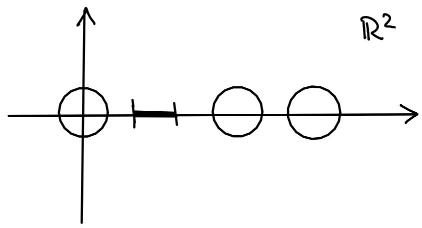


4 2d Open-closed TQFTs

$\ell \in \mathbb{N}_0$, $\underline{n} = (n_0, n_1, \dots, n_{\ell-1}) \in \{0,1\}^\ell$

$$\Sigma_{\underline{n}} := \coprod_{j=0}^{\ell-1} \Sigma^{(n_j)}((j,0)) \subseteq \mathbb{R}^2 \qquad \Sigma^{(0)}(x,y) = \partial B_{1/4}(x,y)$$

$$\Sigma^{(1)}(x,y) = [x-\tfrac{1}{4}, x+\tfrac{1}{4}] \times \{y\}$$

$\Sigma_{(0100)} =$ 

<u>Def:</u> The category $2\mathrm{Cob}^{ext}$ of 2-dimensional open-closed cobordisms consists of

objects: $\underline{n} \in \{0,1\}^\ell$, $\ell \in \mathbb{N}_0$

morphisms: $[(M, \psi_1, \psi_2)] : \underline{n} \longrightarrow \underline{n}'$

M compact oriented 2-dimensional $\langle 2 \rangle$-manifold

$$\left.\begin{array}{l} \psi_1 : \Sigma^*_{\underline{n}} \longrightarrow \psi_1(\Sigma^*_{\underline{n}}) \subseteq \partial_0 M \\ \psi_2 : \Sigma_{\underline{n}'} \longrightarrow \psi_2(\Sigma_{\underline{n}'}) \subseteq \partial_0 M \end{array}\right\} \text{ orientation preserving diffeomorphisms}$$

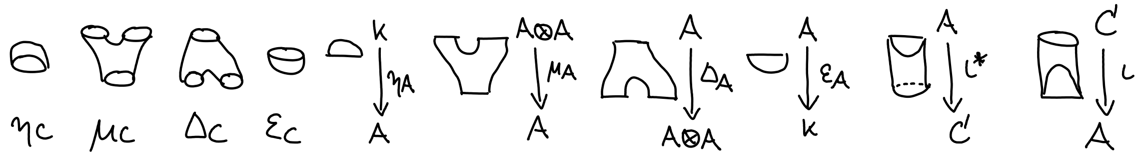
such that

$$\partial_0 M = \psi_1(\Sigma_{n^*}) \amalg \psi_2(\Sigma_{\underline{n}'}).$$

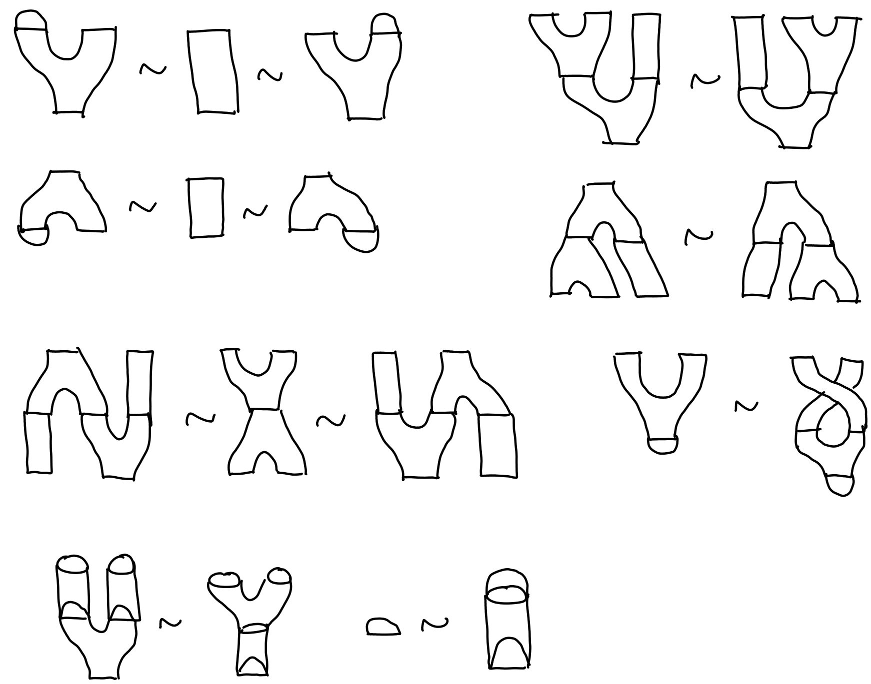
'$\sim$' as above.

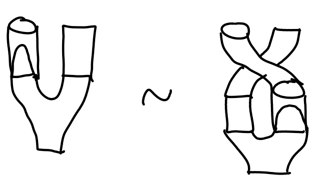
<u>Def:</u> A 2-dimensional <u>open-closed TQFT</u> is a symmetric monoidal functor
$Z : 2\mathrm{Cob}^{ext} \longrightarrow \mathrm{Vect}_K$.

Morse data:

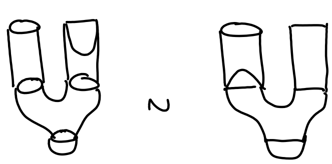


Equivalent morphisms: as above and in addition

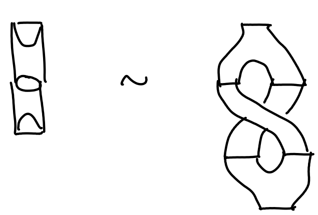




(knowledge)



(duality)



(Cardy condition)

Def: A <u>knowledgeable Frobenius algebra</u> (A, C, ι, ι^*) consists of

(a) a symmetric Frobenius algebra $(A, \mu_A, \eta_A, \Delta_A, \varepsilon_A)$

(b) a commutative Frobenius algebra $(C, \mu_C, \eta_C, \Delta_C, \varepsilon_C)$

(c) linear maps $\iota : C \to A$, $\iota^* : A \to C$

such that

(1) ι is a homomorphism of algebras

(2) $\mu_A \circ (\iota \otimes id_A) = \mu_A \circ \tau \circ (\iota \otimes id_A)$

(3) $\varepsilon_C \circ \mu_C \circ (id_C \otimes \iota^*) = \varepsilon_A \circ \mu_A \circ (\iota \otimes id_A)$

(4) $\iota \circ \iota^* = \mu_A \circ \tau \circ \Delta_A$

Thm: [AL, HP]

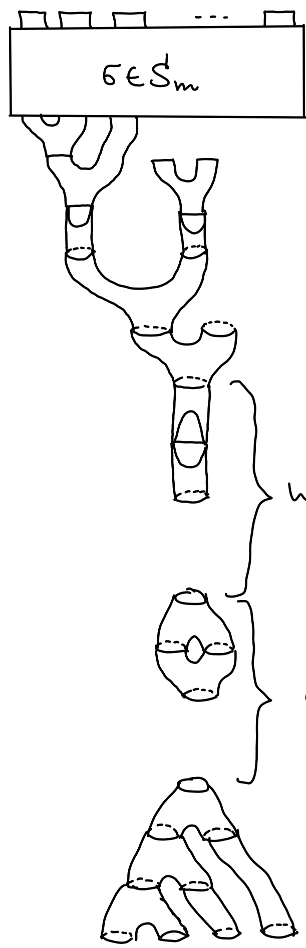
The category $2Cob^{ext}$ is the free symmetric monoidal category generated by a knowledgeable Frobenius algebra object.

$\Rightarrow$ $\quad$ 2d-OCTQFT $:= \mathrm{SymMon}[2Cob^{ext}, Vect_k] \simeq$ KFrob.

proof:
1) Find generators for the morphisms of $2Cob^{ext}$ (Morse theory)
2) relations are necessary, i.e. induced by '$\sim$'
3) relations are sufficient (normal form)
4) invariants: genus, window number, bdry permutation

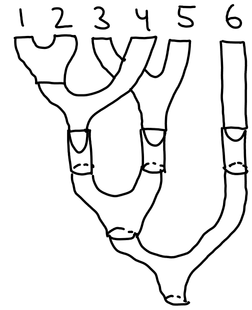
Normal form of an open-closed cobordism

$M:(\underbrace{1,1,\dots,1}_{m}) \longrightarrow (\underbrace{0,0,\dots,0}_{\ell})$



window number

genus



$\sigma = (124)(35) \in S_m$
(boundary permutation)