Combinatorial characterization of fusion categories

Hendryk Pfeiffer

Department of Mathematics
The University of British Columbia
1984 Mathematics Road
Vancouver BC, V6T 1Z2
Canada

July 22, 2010
(XIXth Oporto Meeting, Faro, Portugal)

Quantum 3-manifold invariants

Monoidal categories with extra structure:

- Turaev–Viro: Finitely semisimple spherical category $\mathcal{C}$
- Reshetikhin–Turaev: Modular category $\mathcal{C}$

Two sorts of tensor products:

- If $\mathcal{C} \simeq H-\mathrm{Mod}$, H Hopf algebra: easy tensor product
- Otherwise: difficult tensor product

Observation:

- easy tensor product $\longleftrightarrow$ homotopy type invariant
- difficult tensor product $\longleftrightarrow$ stronger invariant

Quantum 3-manifold invariants

Monoidal categories with extra structure:

- Turaev–Viro: Finitely semisimple spherical category $\mathcal{C}$
- Reshetikhin–Turaev: Modular category $\mathcal{C}$

Two sorts of tensor products:

- If $\mathcal{C} \simeq H - \mathrm{Mod}$, H Hopf algebra: easy tensor product
- Otherwise: difficult tensor product

Observation:

- easy tensor product $\longleftrightarrow$ homotopy type invariant
- difficult tensor product $\longleftrightarrow$ stronger invariant

Quantum 3-manifold invariants

Monoidal categories with extra structure:

- Turaev–Viro: Finitely semisimple spherical category $\mathcal{C}$
- Reshetikhin–Turaev: Modular category $\mathcal{C}$

Two sorts of tensor products:

- If $\mathcal{C} \simeq H - \text{Mod}$, H Hopf algebra: easy tensor product
- Otherwise: difficult tensor product

Observation:

- easy tensor product $\longleftrightarrow$ homotopy type invariant
- difficult tensor product $\longleftrightarrow$ stronger invariant

Quantum 3-manifold invariants

Monoidal categories with extra structure:

- Turaev–Viro: Finitely semisimple spherical category $\mathcal{C}$
- Reshetikhin–Turaev: Modular category $\mathcal{C}$

Two sorts of tensor products:

- If $\mathcal{C} \simeq H - \text{Mod}$, H Hopf algebra: easy tensor product
- Otherwise: difficult tensor product

Observation:

- easy tensor product $\longleftrightarrow$ homotopy type invariant
- difficult tensor product $\longleftrightarrow$ stronger invariant

Quantum 3-manifold invariants

Monoidal categories with extra structure:

- Turaev–Viro: Finitely semisimple spherical category $\mathcal{C}$
- Reshetikhin–Turaev: Modular category $\mathcal{C}$

Two sorts of tensor products:

- If $\mathcal{C} \simeq H - \mathrm{Mod}$, H Hopf algebra: easy tensor product
- Otherwise: difficult tensor product

Observation:

- easy tensor product $\longleftrightarrow$ homotopy type invariant
- difficult tensor product $\longleftrightarrow$ stronger invariant

Quantum 3-manifold invariants

Monoidal categories with extra structure:

- Turaev–Viro: Finitely semisimple spherical category $\mathcal{C}$
- Reshetikhin–Turaev: Modular category $\mathcal{C}$

Two sorts of tensor products:

- If $\mathcal{C} \simeq H - \mathrm{Mod}$, H Hopf algebra: easy tensor product
- Otherwise: difficult tensor product

Observation:

- easy tensor product $\longleftrightarrow$ homotopy type invariant
- difficult tensor product $\longleftrightarrow$ stronger invariant

Quantum 3-manifold invariants

Monoidal categories with extra structure:

- Turaev–Viro: Finitely semisimple spherical category $\mathcal{C}$
- Reshetikhin–Turaev: Modular category $\mathcal{C}$

Two sorts of tensor products:

- If $\mathcal{C} \simeq H-\text{Mod}$, H Hopf algebra: easy tensor product
- Otherwise: difficult tensor product

Observation:

- easy tensor product $\longleftrightarrow$ homotopy type invariant
- difficult tensor product $\longleftrightarrow$ stronger invariant

Quantum 3-manifold invariants

Monoidal categories with extra structure:

- Turaev–Viro: Finitely semisimple spherical category $\mathcal{C}$
- Reshetikhin–Turaev: Modular category $\mathcal{C}$

Two sorts of tensor products:

- If $\mathcal{C} \simeq H - \mathrm{Mod}$, H Hopf algebra: easy tensor product
- Otherwise: difficult tensor product

Observation:

- easy tensor product $\longleftrightarrow$ homotopy type invariant
- difficult tensor product $\longleftrightarrow$ stronger invariant

Quantum 3-manifold invariants

Monoidal categories with extra structure:

- Turaev–Viro: Finitely semisimple spherical category $\mathcal{C}$
- Reshetikhin–Turaev: Modular category $\mathcal{C}$

Two sorts of tensor products:

- If $\mathcal{C} \simeq H - \text{Mod}$, H Hopf algebra: easy tensor product
- Otherwise: difficult tensor product

Observation:

- easy tensor product $\longleftrightarrow$ homotopy type invariant
- difficult tensor product $\longleftrightarrow$ stronger invariant

Outline

This talk

- is about the difficult $\otimes$-products
- applies to all (multi-) fusion categories
- gives a combinatorial description of the $\otimes$-product
- example: modular categories associated with $U_q(\mathfrak{sl}_2)$

Outline

This talk

- is about the difficult $\otimes$-products
- applies to all (multi-) fusion categories
- gives a combinatorial description of the $\otimes$-product
- example: modular categories associated with $U_q(\mathfrak{sl}_2)$

Outline

This talk

- is about the difficult $\otimes$-products
- applies to all (multi-) fusion categories
- gives a combinatorial description of the $\otimes$-product
- example: modular categories associated with $U_q(\mathfrak{sl}_2)$

Outline

This talk

- is about the difficult $\otimes$-products
- applies to all (multi-) fusion categories
- gives a combinatorial description of the $\otimes$-product
- example: modular categories associated with $U_q(\mathfrak{sl}_2)$

Outline

This talk

- is about the difficult $\otimes$-products
- applies to all (multi-) fusion categories
- gives a combinatorial description of the $\otimes$-product
- example: modular categories associated with $U_q(\mathfrak{sl}_2)$

Recall $SL_q(2)$

$SL_q(2) = \mathbb{C}\{\, t_{ij} \mid 1 \le i, j \le 2 \,\}/(\text{relations})$

- quadratic relations $RTT - TTR$

$$\sum_{k,\ell} \Big(R^{k\ell}_{ij} t^p_k t^q_\ell - t^k_i t^\ell_j R^{pq}_{k\ell} \Big), \quad R = \begin{pmatrix} q & & & \\ & 0 & 1 & \\ & 1 & q - q^{-1} & \\ & & & q \end{pmatrix}$$

- inhomogeneous relation (degree 0 and 2)

$$1 - \mathrm{qdet}, \qquad \mathrm{qdet} = da - qbc$$

- $\Longrightarrow \mathcal{M}^{SL_q(2)} \simeq {}_{U_q(\mathfrak{sl}_2)}\mathcal{M}$

We will get an analogous description of (multi-)fusion categories.

Recall $SL_q(2)$

$SL_q(2) = \mathbb{C}\{\, t_{ij} \mid 1 \le i,j \le 2 \,\}/(\text{relations})$

- quadratic relations $RTT - TTR$

$$\sum_{k,\ell} \left(R_{ij}^{k\ell} t_k^p t_\ell^q - t_i^k t_j^\ell R_{k\ell}^{pq} \right), \quad R = \begin{pmatrix} q & & & \\ & 0 & 1 & \\ & 1 & q - q^{-1} & \\ & & & q \end{pmatrix}$$

- inhomogeneous relation (degree 0 and 2)

$$1 - \text{qdet}, \qquad \text{qdet} = da - qbc$$

- $\implies \mathcal{M}^{SL_q(2)} \simeq {}_{U_q(\mathfrak{sl}_2)}\mathcal{M}$

We will get an analogous description of (multi-)fusion categories.

Recall $SL_q(2)$

$SL_q(2) = \mathbb{C}\{\, t_{ij} \mid 1 \le i,j \le 2\,\}/(\text{relations})$

- quadratic relations $RTT - TTR$

$$\sum_{k,\ell}\left(R^{k\ell}_{ij} t^p_k t^q_\ell - t^k_i t^\ell_j R^{pq}_{k\ell}\right), \quad R = \begin{pmatrix} q & & & \\ & 0 & 1 & \\ & 1 & q - q^{-1} & \\ & & & q \end{pmatrix}$$

- inhomogeneous relation (degree 0 and 2)

$$1 - \mathrm{qdet}, \qquad \mathrm{qdet} = da - qbc$$

- $\Longrightarrow \mathcal{M}^{SL_q(2)} \simeq {}_{U_q(\mathfrak{sl}_2)}\mathcal{M}$

We will get an analogous description of (multi-)fusion categories.

Recall $SL_q(2)$

$SL_q(2) = \mathbb{C}\{\, t_{ij} \mid 1 \le i, j \le 2 \,\}/(\text{relations})$

- quadratic relations $RTT - TTR$

$$\sum_{k,\ell} \left(R_{ij}^{k\ell} t_k^p t_\ell^q - t_i^k t_j^\ell R_{k\ell}^{pq} \right), \quad R = \begin{pmatrix} q & & & \\ & 0 & 1 & \\ & 1 & q - q^{-1} & \\ & & & q \end{pmatrix}$$

- inhomogeneous relation (degree 0 and 2)

$$1 - \mathrm{qdet}, \qquad \mathrm{qdet} = da - qbc$$

- $\implies \mathcal{M}^{SL_q(2)} \simeq {}_{U_q(\mathfrak{sl}_2)}\mathcal{M}$

We will get an analogous description of (multi-)fusion categories.

Recall $SL_q(2)$

$SL_q(2) = \mathbb{C}\{\, t_{ij} \mid 1 \le i,j \le 2 \,\}/(\text{relations})$

- quadratic relations $RTT - TTR$

$$\sum_{k,\ell} \left(R^{k\ell}_{ij} t^p_k t^q_\ell - t^k_i t^\ell_j R^{pq}_{k\ell} \right), \quad R = \begin{pmatrix} q & & & \\ & 0 & 1 & \\ & 1 & q - q^{-1} & \\ & & & q \end{pmatrix}$$

- inhomogeneous relation (degree 0 and 2)

$$1 - \text{qdet}, \qquad \text{qdet} = da - qbc$$

- $\Longrightarrow \mathcal{M}^{SL_q(2)} \simeq {}_{U_q(\mathfrak{sl}_2)}\mathcal{M}$

We will get an analogous description of (multi-)fusion categories.

Fusion categories

Definition

A multi-fusion category $\mathcal{C}$

- is monoidal $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \rho, \lambda)$
- is k-linear (k field), additive (all finite biproducts)
- has $\mathrm{Hom}(X, Y)$ finite-dimensional over k for all $X, Y \in |\mathcal{C}|$
- is autonomous (each object is equipped with a left-dual)
- is finitely semisimple
- has $\mathrm{End}(X) \cong k$ for all simple $X \in |\mathcal{C}|$
- $\implies$ is abelian

Fusion categories

Definition

A multi-fusion category $\mathcal{C}$

- is monoidal $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \rho, \lambda)$
- is k-linear (k field), additive (all finite biproducts)
- has $\mathrm{Hom}(X, Y)$ finite-dimensional over k for all $X, Y \in |\mathcal{C}|$
- is autonomous (each object is equipped with a left-dual)
- is finitely semisimple
- has $\mathrm{End}(X) \cong k$ for all simple $X \in |\mathcal{C}|$
- $\Longrightarrow$ is abelian

Fusion categories

Definition

A multi-fusion category $\mathcal{C}$

- is monoidal $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \rho, \lambda)$
- is k-linear (k field), additive (all finite biproducts)
- has $\mathrm{Hom}(X, Y)$ finite-dimensional over k for all $X, Y \in |\mathcal{C}|$
- is autonomous (each object is equipped with a left-dual)
- is finitely semisimple
- has $\mathrm{End}(X) \cong k$ for all simple $X \in |\mathcal{C}|$
- $\Longrightarrow$ is abelian

Fusion categories

Definition

A multi-fusion category $\mathcal{C}$

- is monoidal $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \rho, \lambda)$
- is k-linear (k field), additive (all finite biproducts)
- has $\mathrm{Hom}(X, Y)$ finite-dimensional over k for all $X, Y \in |\mathcal{C}|$
- is autonomous (each object is equipped with a left-dual)
- is finitely semisimple
- has $\mathrm{End}(X) \cong k$ for all simple $X \in |\mathcal{C}|$
- $\Longrightarrow$ is abelian

Fusion categories

Definition

A multi-fusion category $\mathcal{C}$

- is monoidal $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \rho, \lambda)$
- is k-linear (k field), additive (all finite biproducts)
- has $\mathrm{Hom}(X, Y)$ finite-dimensional over k for all $X, Y \in |\mathcal{C}|$
- is autonomous (each object is equipped with a left-dual)
- is finitely semisimple
- has $\mathrm{End}(X) \cong k$ for all simple $X \in |\mathcal{C}|$
- $\Longrightarrow$ is abelian

Fusion categories

Definition

A multi-fusion category $\mathcal{C}$

- is monoidal $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \rho, \lambda)$
- is k-linear (k field), additive (all finite biproducts)
- has $\mathrm{Hom}(X, Y)$ finite-dimensional over k for all $X, Y \in |\mathcal{C}|$
- is autonomous (each object is equipped with a left-dual)
- is finitely semisimple
- has $\mathrm{End}(X) \cong k$ for all simple $X \in |\mathcal{C}|$
- $\Longrightarrow$ is abelian

Fusion categories

Definition

A multi-fusion category $\mathcal{C}$

- is monoidal $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \rho, \lambda)$
- is k-linear (k field), additive (all finite biproducts)
- has $\mathrm{Hom}(X, Y)$ finite-dimensional over k for all $X, Y \in |\mathcal{C}|$
- is autonomous (each object is equipped with a left-dual)
- is finitely semisimple
- has $\mathrm{End}(X) \cong k$ for all simple $X \in |\mathcal{C}|$
- $\Longrightarrow$ is abelian

Fusion categories

Definition

A multi-fusion category $\mathcal{C}$

- is monoidal $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \rho, \lambda)$
- is k-linear (k field), additive (all finite biproducts)
- has $\mathrm{Hom}(X, Y)$ finite-dimensional over k for all $X, Y \in |\mathcal{C}|$
- is autonomous (each object is equipped with a left-dual)
- is finitely semisimple
- has $\mathrm{End}(X) \cong k$ for all simple $X \in |\mathcal{C}|$
- $\Longrightarrow$ is abelian

Canonical forgetful functor

Theorem (Hayashi, Hai)

$\mathcal{C}$ (multi-)fusion category.

- Small projective generator $\hat{V} = \bigoplus_{j \in J} V_j$
- The *long canonical forgetful functor*

$$\omega \colon \mathcal{C} \to \mathbf{Vect}_k, \qquad X \mapsto \mathrm{Hom}(\hat{V}, \hat{V} \otimes X)$$

  is k-linear, faithful and exact.

Theorem (classical)

$H = \mathbf{coend}(\mathcal{C}, \omega) = \bigoplus_{j \in J} (\omega V_j)^* \otimes \omega V_j$

- is a coassociative counital coalgebra
- is finite-dimensional and split cosemisimple
- $\mathcal{M}^H \simeq \mathcal{C}$ as k-linear additive categories

Canonical forgetful functor

Theorem (Hayashi, Hai)

$\mathcal{C}$ (multi-)fusion category.

- Small projective generator $\hat{V} = \bigoplus_{j \in J} V_j$
- The *long canonical forgetful functor*

$$\omega \colon \mathcal{C} \to \mathbf{Vect}_k, \qquad X \mapsto \mathrm{Hom}(\hat{V}, \hat{V} \otimes X)$$

  is k-linear, faithful and exact.

Theorem (classical)

$$H = \mathbf{coend}(\mathcal{C}, \omega) = \bigoplus_{j \in J} (\omega V_j)^* \otimes \omega V_j$$

- is a coassociative counital coalgebra
- is finite-dimensional and split cosemisimple
- $\mathcal{M}^H \simeq \mathcal{C}$ as k-linear additive categories

Canonical forgetful functor

Theorem (Hayashi, Hai)

$\mathcal{C}$ (multi-)fusion category.

- Small projective generator $\hat{V} = \bigoplus_{j \in J} V_j$
- The *long canonical forgetful functor*

$$\omega \colon \mathcal{C} \to \mathbf{Vect}_k, \qquad X \mapsto \mathrm{Hom}(\hat{V}, \hat{V} \otimes X)$$

  is k-linear, faithful and exact.

Theorem (classical)

$H = \mathbf{coend}(\mathcal{C}, \omega) = \bigoplus_{j \in J} (\omega V_j)^* \otimes \omega V_j$

- is a coassociative counital coalgebra
- is finite-dimensional and split cosemisimple
- $\mathcal{M}^H \simeq \mathcal{C}$ as k-linear additive categories

Canonical forgetful functor

Theorem (Hayashi, Hai)

$\mathcal{C}$ (multi-)fusion category.

- Small projective generator $\hat{V} = \bigoplus_{j \in J} V_j$
- The *long canonical forgetful functor*

$$\omega\colon \mathcal{C} \to \mathbf{Vect}_k, \qquad X \mapsto \mathrm{Hom}(\hat{V}, \hat{V} \otimes X)$$

is k-linear, faithful and exact.

Theorem (classical)

$H = \mathbf{coend}(\mathcal{C}, \omega) = \bigoplus_{j \in J} (\omega V_j)^* \otimes \omega V_j$

- is a coassociative counital coalgebra
- is finite-dimensional and split cosemisimple
- $\mathcal{M}^H \simeq \mathcal{C}$ as k-linear additive categories

Canonical forgetful functor

Theorem (Hayashi, Hai)

$\mathcal{C}$ (multi-)fusion category.

- Small projective generator $\hat{V} = \bigoplus_{j\in J} V_j$
- The *long canonical forgetful functor*

$$\omega\colon \mathcal{C} \to \mathbf{Vect}_k, \qquad X \mapsto \mathrm{Hom}(\hat{V}, \hat{V} \otimes X)$$

  is k-linear, faithful and exact.

Theorem (classical)

$H = \mathbf{coend}(\mathcal{C}, \omega) = \bigoplus_{j\in J} (\omega V_j)^* \otimes \omega V_j$

- is a coassociative counital coalgebra
- is finite-dimensional and split cosemisimple
- $\mathcal{M}^H \simeq \mathcal{C}$ as k-linear additive categories

Canonical forgetful functor

Theorem (Hayashi, Hai)

$\mathcal{C}$ (multi-)fusion category.

- Small projective generator $\hat{V} = \bigoplus_{j\in J} V_j$
- The *long canonical forgetful functor*

$$\omega\colon \mathcal{C} \to \mathbf{Vect}_k, \qquad X \mapsto \mathrm{Hom}(\hat{V}, \hat{V} \otimes X)$$

is k-linear, faithful and exact.

Theorem (classical)

$$H = \mathbf{coend}(\mathcal{C}, \omega) = \bigoplus_{j\in J} (\omega V_j)^* \otimes \omega V_j$$

- is a coassociative counital coalgebra
- is finite-dimensional and split cosemisimple
- $\mathcal{M}^H \simeq \mathcal{C}$ as k-linear additive categories

Canonical forgetful functor

Theorem (Hayashi, Hai)

$\mathcal{C}$ (multi-)fusion category.

- Small projective generator $\hat{V} = \bigoplus_{j \in J} V_j$
- The *long canonical forgetful functor*

$$\omega \colon \mathcal{C} \to \mathbf{Vect}_k, \qquad X \mapsto \mathrm{Hom}(\hat{V}, \hat{V} \otimes X)$$

  is k-linear, faithful and exact.

Theorem (classical)

$$H = \mathbf{coend}(\mathcal{C}, \omega) = \bigoplus_{j \in J} (\omega V_j)^* \otimes \omega V_j$$

- is a coassociative counital coalgebra
- is finite-dimensional and split cosemisimple
- $\mathcal{M}^H \simeq \mathcal{C}$ as k-linear additive categories

Weak Hopf Algebra structure

Theorem (Szlachányi, HP)

$\mathcal{C}$ (multi-)fusion category. Then $\omega\colon \mathcal{C} \to \mathbf{Vect}_k$

- is lax and oplax monoidal
- has a separable Frobenius structure

Theorem (Hayashi, Hai, Ostrik, HP)

$H = \mathbf{coend}(\mathcal{C}, \omega)$

- is a Weak Hopf Algebra
- has commutative base algebras
- $H \cong H^*$ as Weak Hopf Algebras
- $\mathcal{M}^H \simeq \mathcal{C}$ as k-linear additive autonomous categories

Weak Hopf Algebra structure

Theorem (Szlachányi, HP)

$\mathcal{C}$ (multi-)fusion category. Then $\omega\colon \mathcal{C} \to \mathbf{Vect}_k$

- is lax and oplax monoidal
- has a separable Frobenius structure

Theorem (Hayashi, Hai, Ostrik, HP)

$H = \mathbf{coend}(\mathcal{C}, \omega)$

- is a Weak Hopf Algebra
- has commutative base algebras
- $H \cong H^*$ as Weak Hopf Algebras
- $\mathcal{M}^H \simeq \mathcal{C}$ as k-linear additive autonomous categories

Weak Hopf Algebra structure

Theorem (Szlachányi, HP)

$\mathcal{C}$ (multi-)fusion category. Then $\omega \colon \mathcal{C} \to \mathbf{Vect}_k$

- is lax and oplax monoidal
- has a separable Frobenius structure

Theorem (Hayashi, Hai, Ostrik, HP)

$H = \mathbf{coend}(\mathcal{C}, \omega)$

- is a Weak Hopf Algebra
- has commutative base algebras
- $H \simeq H^*$ as Weak Hopf Algebras
- $\mathcal{M}^H \simeq \mathcal{C}$ as k-linear additive autonomous categories

Weak Hopf Algebra structure

Theorem (Szlachányi, HP)

$\mathcal{C}$ (multi-)fusion category. Then $\omega\colon \mathcal{C} \to \mathbf{Vect}_k$

- is lax and oplax monoidal
- has a separable Frobenius structure

Theorem (Hayashi, Hai, Ostrik, HP)

$H = \mathbf{coend}(\mathcal{C}, \omega)$

- is a Weak Hopf Algebra
- has commutative base algebras
- $H \cong H^*$ as Weak Hopf Algebras
- $\mathcal{M}^H \simeq \mathcal{C}$ as k-linear additive autonomous categories

Weak Hopf Algebra structure

Theorem (Szlachányi, HP)

$\mathcal{C}$ (multi-)fusion category. Then $\omega\colon \mathcal{C} \to \mathbf{Vect}_k$

- is lax and oplax monoidal
- has a separable Frobenius structure

Theorem (Hayashi, Hai, Ostrik, HP)

$H = \mathbf{coend}(\mathcal{C}, \omega)$

- is a Weak Hopf Algebra
- has commutative base algebras
- $H \cong H^*$ as Weak Hopf Algebras
- $\mathcal{M}^H \simeq \mathcal{C}$ as k-linear additive autonomous categories

Weak Hopf Algebra structure

Theorem (Szlachányi, HP)

$\mathcal{C}$ (multi-)fusion category. Then $\omega\colon \mathcal{C} \to \mathbf{Vect}_k$

- is lax and oplax monoidal
- has a separable Frobenius structure

Theorem (Hayashi, Hai, Ostrik, HP)

$H = \mathbf{coend}(\mathcal{C}, \omega)$

- is a Weak Hopf Algebra
- has commutative base algebras
- $H \cong H^*$ as Weak Hopf Algebras
- $\mathcal{M}^H \simeq \mathcal{C}$ as k-linear additive autonomous categories

Weak Hopf Algebra structure

Theorem (Szlachányi, HP)

$\mathcal{C}$ (multi-)fusion category. Then $\omega\colon \mathcal{C} \to \mathbf{Vect}_k$

- is lax and oplax monoidal
- has a separable Frobenius structure

Theorem (Hayashi, Hai, Ostrik, HP)

$H = \mathbf{coend}(\mathcal{C}, \omega)$

- is a Weak Hopf Algebra
- has commutative base algebras
- $H \cong H^*$ as Weak Hopf Algebras
- $\mathcal{M}^H \simeq \mathcal{C}$ as k-linear additive autonomous categories

Weak Hopf Algebra structure

Theorem (Szlachányi, HP)

$\mathcal{C}$ (multi-)fusion category. Then $\omega \colon \mathcal{C} \to \mathbf{Vect}_k$

- is lax and oplax monoidal
- has a separable Frobenius structure

Theorem (Hayashi, Hai, Ostrik, HP)

$H = \mathbf{coend}(\mathcal{C}, \omega)$

- is a Weak Hopf Algebra
- has commutative base algebras
- $H \cong H^*$ as Weak Hopf Algebras
- $\mathcal{M}^H \simeq \mathcal{C}$ as k-linear additive autonomous categories

The Dimension Graph

Definition

$\mathcal{C}$ (multi-)fusion, M monoidal generator of $\mathcal{C}$.

The finite directed *dimension graph* $\mathcal{G}$ of $\mathcal{C}$:

- vertices: J (classes of simple objects)
- edges $j \to \ell$: basis of $\mathrm{Hom}(V_j, V_\ell \otimes M)$

Notation

- $\mathcal{G}^0$: vertices
- $\mathcal{G}^m$: paths of length $m \geq 1$

The Dimension Graph

Definition

$\mathcal{C}$ (multi-)fusion, M monoidal generator of $\mathcal{C}$.

The finite directed *dimension graph* $\mathcal{G}$ of $\mathcal{C}$:

- vertices: J (classes of simple objects)
- edges $j \to \ell$: basis of $\mathrm{Hom}(V_j, V_\ell \otimes M)$

Notation

- $\mathcal{G}^0$: vertices
- $\mathcal{G}^m$: paths of length $m \geq 1$

The Dimension Graph

Definition

$\mathcal{C}$ (multi-)fusion, M monoidal generator of $\mathcal{C}$.

The finite directed *dimension graph* $\mathcal{G}$ of $\mathcal{C}$:

- vertices: J (classes of simple objects)
- edges $j \to \ell$: basis of $\mathrm{Hom}(V_j, V_\ell \otimes M)$

Notation

- $\mathcal{G}^0$: vertices
- $\mathcal{G}^m$: paths of length $m \geq 1$

The Dimension Graph

Definition

$\mathcal{C}$ (multi-)fusion, M monoidal generator of $\mathcal{C}$.

The finite directed *dimension graph* $\mathcal{G}$ of $\mathcal{C}$:

- vertices: J (classes of simple objects)
- edges $j \to \ell$: basis of $\mathrm{Hom}(V_j, V_\ell \otimes M)$

Notation

- $\mathcal{G}^0$: vertices
- $\mathcal{G}^m$: paths of length $m \geq 1$

The Dimension Graph

Definition

$\mathcal{C}$ (multi-)fusion, M monoidal generator of $\mathcal{C}$.

The finite directed *dimension graph* $\mathcal{G}$ of $\mathcal{C}$:

- vertices: J (classes of simple objects)
- edges $j \to \ell$: basis of $\mathrm{Hom}(V_j, V_\ell \otimes M)$

Notation

- $\mathcal{G}^0$: vertices
- $\mathcal{G}^m$: paths of length $m \geq 1$

The Dimension Graph

Definition

$\mathcal{C}$ (multi-)fusion, M monoidal generator of $\mathcal{C}$.

The finite directed *dimension graph* $\mathcal{G}$ of $\mathcal{C}$:

- vertices: J (classes of simple objects)
- edges $j \to \ell$: basis of $\mathrm{Hom}(V_j, V_\ell \otimes M)$

Notation

- $\mathcal{G}^0$: vertices
- $\mathcal{G}^m$: paths of length $m \geq 1$

Modular categories associated with $U_q(\mathfrak{sl}_2)$

Example

- $p = 3, 4, 5, \ldots$
- $\zeta^{4p} = 1$ (primitive $4p$-th root of unity)
- Field $k = \mathbb{C}$ or $k = \mathbb{Q}(\zeta)$
- Dimension graph:

Modular categories associated with $U_q(\mathfrak{sl}_2)$

Example

- $p = 3, 4, 5, \ldots$
- $\zeta^{4p} = 1$ (primitive $4p$-th root of unity)
- Field $k = \mathbb{C}$ or $k = \mathbb{Q}(\zeta)$
- Dimension graph:

Modular categories associated with $U_q(\mathfrak{sl}_2)$

Example

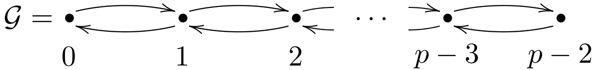
- $p = 3, 4, 5, \ldots$
- $\zeta^{4p} = 1$ (primitive $4p$-th root of unity)
- Field $k = \mathbb{C}$ or $k = \mathbb{Q}(\zeta)$
- Dimension graph:

$\mathcal{G} =$

Modular categories associated with $U_q(\mathfrak{sl}_2)$

Example

- $p = 3, 4, 5, \ldots$
- $\zeta^{4p} = 1$ (primitive $4p$-th root of unity)
- Field $k = \mathbb{C}$ or $k = \mathbb{Q}(\zeta)$
- Dimension graph:



The Path Algebra of $\mathcal{G} \times \mathcal{G}$

The path algebra of $\mathcal{G} \times \mathcal{G}$

$$H[\mathcal{G}] = k(\mathcal{G} \times \mathcal{G}) = \bigoplus_{m \geq 0} \underbrace{k\mathcal{G}^m \otimes k\mathcal{G}^m}_{H[\mathcal{G}]_m}$$

- is a graded associative unital algebra over k
- $p, q \in \mathcal{G}^m$: $[p|q]_m = p \otimes q \in H[\mathcal{G}]_m$ (basis elements)
- is a direct sum of matrix coalgebras
- is a split cosemisimple coassociative counital coalgebra

Theorem

$H[\mathcal{G}]$ is a graded, split cosemisimple Weak Bialgebra.

The Path Algebra of $\mathcal{G} \times \mathcal{G}$

The path algebra of $\mathcal{G} \times \mathcal{G}$

$$H[\mathcal{G}] = k(\mathcal{G} \times \mathcal{G}) = \bigoplus_{m \geq 0} \underbrace{k\mathcal{G}^m \otimes k\mathcal{G}^m}_{H[\mathcal{G}]_m}$$

- is a graded associative unital algebra over k
- $p, q \in \mathcal{G}^m$, $[p|q]_m = p \otimes q \in H[\mathcal{G}]_m$ (basis elements)
- is a direct sum of matrix coalgebras
- is a split cosemisimple coassociative counital coalgebra

Theorem

$H[\mathcal{G}]$ is a graded, split cosemisimple Weak Bialgebra.

The Path Algebra of $\mathcal{G} \times \mathcal{G}$

The path algebra of $\mathcal{G} \times \mathcal{G}$

$$H[\mathcal{G}] = k(\mathcal{G} \times \mathcal{G}) = \bigoplus_{m \geq 0} \underbrace{k\mathcal{G}^m \otimes k\mathcal{G}^m}_{H[\mathcal{G}]_m}$$

- is a graded associative unital algebra over k
- $p, q \in \mathcal{G}^m$, $[p|q]_m = p \otimes q \in H[\mathcal{G}]_m$ (basis elements)
- is a direct sum of matrix coalgebras
- is a split cosemisimple coassociative counital coalgebra

Theorem
$H[\mathcal{G}]$ is a graded, split cosemisimple Weak Bialgebra.

The Path Algebra of $\mathcal{G} \times \mathcal{G}$

The path algebra of $\mathcal{G} \times \mathcal{G}$

$$H[\mathcal{G}] = k(\mathcal{G} \times \mathcal{G}) = \bigoplus_{m \geq 0} \underbrace{k\mathcal{G}^m \otimes k\mathcal{G}^m}_{H[\mathcal{G}]_m}$$

- is a graded associative unital algebra over k
- $p, q \in \mathcal{G}^m$, $[p|q]_m = p \otimes q \in H[\mathcal{G}]_m$ (basis elements)
- is a direct sum of matrix coalgebras
- is a split cosemisimple coassociative counital coalgebra

Theorem

$H[\mathcal{G}]$ is a graded, split cosemisimple Weak Bialgebra.

The Path Algebra of $\mathcal{G} \times \mathcal{G}$

The path algebra of $\mathcal{G} \times \mathcal{G}$

$$H[\mathcal{G}] = k(\mathcal{G} \times \mathcal{G}) = \bigoplus_{m \geq 0} \underbrace{(k\mathcal{G}^m)^* \otimes k\mathcal{G}^m}_{H[\mathcal{G}]_m}$$

- is a graded associative unital algebra over k
- $p, q \in \mathcal{G}^m$, $[p|q]_m = p \otimes q \in H[\mathcal{G}]_m$ (basis elements)
- is a direct sum of matrix coalgebras
- is a split cosemisimple coassociative counital coalgebra

Theorem
$H[\mathcal{G}]$ is a graded, split cosemisimple Weak Bialgebra.

The Path Algebra of $\mathcal{G} \times \mathcal{G}$

The path algebra of $\mathcal{G} \times \mathcal{G}$

$$H[\mathcal{G}] = k(\mathcal{G} \times \mathcal{G}) = \bigoplus_{m \geq 0} \underbrace{(k\mathcal{G}^m)^* \otimes k\mathcal{G}^m}_{H[\mathcal{G}]_m}$$

- is a graded associative unital algebra over k
- $p, q \in \mathcal{G}^m$, $[p|q]_m = p \otimes q \in H[\mathcal{G}]_m$ (basis elements)
- is a direct sum of matrix coalgebras
- is a split cosemisimple coassociative counital coalgebra

Theorem

$H[\mathcal{G}]$ is a graded, split cosemisimple Weak Bialgebra.

The Path Algebra of $\mathcal{G} \times \mathcal{G}$

The path algebra of $\mathcal{G} \times \mathcal{G}$

$$H[\mathcal{G}] = k(\mathcal{G} \times \mathcal{G}) = \bigoplus_{m \geq 0} \underbrace{(k\mathcal{G}^m)^* \otimes k\mathcal{G}^m}_{H[\mathcal{G}]_m}$$

- is a graded associative unital algebra over k
- $p, q \in \mathcal{G}^m$, $[p|q]_m = p \otimes q \in H[\mathcal{G}]_m$ (basis elements)
- is a direct sum of matrix coalgebras
- is a split cosemisimple coassociative counital coalgebra

Theorem

$H[\mathcal{G}]$ is a graded, split cosemisimple Weak Bialgebra.

The Path Algebra of $\mathcal{G} \times \mathcal{G}$

The path algebra of $\mathcal{G} \times \mathcal{G}$

$$H[\mathcal{G}] = k(\mathcal{G} \times \mathcal{G}) = \bigoplus_{m \geq 0} \underbrace{(k\mathcal{G}^m)^* \otimes k\mathcal{G}^m}_{H[\mathcal{G}]_m}$$

- is a graded associative unital algebra over k
- $p, q \in \mathcal{G}^m$, $[p|q]_m = p \otimes q \in H[\mathcal{G}]_m$ (basis elements)
- is a direct sum of matrix coalgebras
- is a split cosemisimple coassociative counital coalgebra

Theorem

$H[\mathcal{G}]$ is a graded, split cosemisimple Weak Bialgebra.

The Weak Bialgebra $H[\mathcal{G}]$

Corepresentation theory of $H[\mathcal{G}] = k(\mathcal{G} \times \mathcal{G})$:

- Simple comodules: $k\mathcal{G}^m$, $m \geq 0$ (split simple)

$$\beta \colon k\mathcal{G}^m \to k\mathcal{G}^m \otimes H[\mathcal{G}], \qquad p \mapsto \sum_{q \in \mathcal{G}^m} q \otimes [q|p]_m$$

- Tensor products:

$$k\mathcal{G}^m \otimes k\mathcal{G}^\ell \cong k\mathcal{G}^{m+\ell} \subseteq k\mathcal{G}^m \otimes_k k\mathcal{G}^\ell$$

(truncated tensor product)

- $\longrightarrow$ Composability of paths in $\mathcal{G} \times \mathcal{G}$ rules the $\otimes$-product

The Weak Bialgebra $H[\mathcal{G}]$

Corepresentation theory of $H[\mathcal{G}] = k(\mathcal{G} \times \mathcal{G})$:

- Simple comodules: $k\mathcal{G}^m$, $m \geq 0$ (split simple)

$$\beta\colon k\mathcal{G}^m \to k\mathcal{G}^m \otimes H[\mathcal{G}], \qquad p \mapsto \sum_{q \in \mathcal{G}^m} q \otimes [q|p]_m$$

- Tensor products:

$$k\mathcal{G}^m \otimes k\mathcal{G}^\ell \cong k\mathcal{G}^{m+\ell} \quad \subseteq \quad k\mathcal{G}^m \otimes_k k\mathcal{G}^\ell$$

(truncated tensor product)

- $\longrightarrow$ Composability of paths in $\mathcal{G} \times \mathcal{G}$ rules the $\otimes$-product

The Weak Bialgebra $H[\mathcal{G}]$

Corepresentation theory of $H[\mathcal{G}] = k(\mathcal{G} \times \mathcal{G})$:

- Simple comodules: $k\mathcal{G}^m$, $m \geq 0$ (split simple)

$$\beta\colon k\mathcal{G}^m \to k\mathcal{G}^m \otimes H[\mathcal{G}], \qquad p \mapsto \sum_{q \in \mathcal{G}^m} q \otimes [q|p]_m$$

- Tensor products:

$$k\mathcal{G}^m \otimes k\mathcal{G}^\ell \cong k\mathcal{G}^{m+\ell} \quad \subseteq \quad k\mathcal{G}^m \otimes_k k\mathcal{G}^\ell$$

(truncated tensor product)

- $\longrightarrow$ Composability of paths in $\mathcal{G} \times \mathcal{G}$ rules the $\otimes$-product

The Weak Bialgebra $H[\mathcal{G}]$

Corepresentation theory of $H[\mathcal{G}] = k(\mathcal{G} \times \mathcal{G})$:

- Simple comodules: $k\mathcal{G}^m$, $m \geq 0$ (split simple)

$$\beta\colon k\mathcal{G}^m \to k\mathcal{G}^m \otimes H[\mathcal{G}], \qquad p \mapsto \sum_{q \in \mathcal{G}^m} q \otimes [q|p]_m$$

- Tensor products:

$$k\mathcal{G}^m \otimes k\mathcal{G}^\ell \cong k\mathcal{G}^{m+\ell} \quad \subseteq \quad k\mathcal{G}^m \otimes_k k\mathcal{G}^\ell$$

(truncated tensor product)

- $\longrightarrow$ Composability of paths in $\mathcal{G} \times \mathcal{G}$ rules the $\otimes$-product

The Weak Bialgebra $H[\mathcal{G}]$

Corepresentation theory of $H[\mathcal{G}] = k(\mathcal{G} \times \mathcal{G})$:

- Simple comodules: $k\mathcal{G}^m$, $m \geq 0$ (split simple)

$$\beta\colon k\mathcal{G}^m \to k\mathcal{G}^m \otimes H[\mathcal{G}], \qquad p \mapsto \sum_{q \in \mathcal{G}^m} q \otimes [q|p]_m$$

- Tensor products:

$$k\mathcal{G}^m \otimes k\mathcal{G}^\ell \cong k\mathcal{G}^{m+\ell} \quad \subseteq \quad k\mathcal{G}^m \otimes_k k\mathcal{G}^\ell$$

(truncated tensor product)

- $\longrightarrow$ Composability of paths in $\mathcal{G} \times \mathcal{G}$ rules the $\otimes$-product

The Weak Bialgebra $H[\mathcal{G}]$

Corepresentation theory of $H[\mathcal{G}] = k(\mathcal{G} \times \mathcal{G})$:

- Simple comodules: $k\mathcal{G}^m$, $m \geq 0$ (split simple)

$$\beta\colon k\mathcal{G}^m \to k\mathcal{G}^m \otimes H[\mathcal{G}], \qquad p \mapsto \sum_{q \in \mathcal{G}^m} q \otimes [q|p]_m$$

- Tensor products:

$$k\mathcal{G}^m \otimes k\mathcal{G}^\ell \cong k\mathcal{G}^{m+\ell} \quad \subseteq \quad k\mathcal{G}^m \otimes_k k\mathcal{G}^\ell$$

  (truncated tensor product)

- $\longrightarrow$ Composability of paths in $\mathcal{G} \times \mathcal{G}$ rules the $\otimes$-product

$H[\mathcal{G}]$ Rules all Fusion Categories

Theorem

There is a surjection of Weak Bialgebras

$$\pi\colon H[\mathcal{G}] \to \mathbf{coend}(\mathcal{C}, \omega)$$

such that

- $\pi((k\mathcal{G}^0)^* \otimes k\mathcal{G}^0) = (\omega\mathbb{1})^* \otimes \omega\mathbb{1}$
- $\pi((k\mathcal{G}^1)^* \otimes k\mathcal{G}^1) = (\omega M)^* \otimes \omega M$

and for the comodules

- $k\mathcal{G}^m$ pushes forward to $(\omega M)^{\otimes m}$

Next Step

Compute the kernel $I = \ker \pi$ and obtain a characterization

$$H[\mathcal{G}]/I \cong \mathbf{coend}(\mathcal{C}, \omega) \quad \text{and} \quad \mathcal{M}^{H[\mathcal{G}]/I} \simeq \mathcal{C}$$

$H[\mathcal{G}]$ Rules all Fusion Categories

Theorem

There is a surjection of Weak Bialgebras

$$\pi\colon H[\mathcal{G}] \to \mathbf{coend}(\mathcal{C}, \omega)$$

such that

- $\pi((k\mathcal{G}^0)^* \otimes k\mathcal{G}^0) = (\omega \mathbb{1})^* \otimes \omega \mathbb{1}$
- $\pi((k\mathcal{G}^1)^* \otimes k\mathcal{G}^1) = (\omega M)^* \otimes \omega M$

and for the comodules

- $k\mathcal{G}^m$ pushes forward to $(\omega M)^{\otimes m}$

Next Step

Compute the kernel $I = \ker \pi$ and obtain a characterization

$$H[\mathcal{G}]/I \cong \mathbf{coend}(\mathcal{C}, \omega) \quad \text{and} \quad \mathcal{M}^{H[\mathcal{G}]/I} \simeq \mathcal{C}$$

$H[\mathcal{G}]$ Rules all Fusion Categories

Theorem

There is a surjection of Weak Bialgebras

$$\pi\colon H[\mathcal{G}] \to \mathbf{coend}(\mathcal{C}, \omega)$$

such that

- $\pi((k\mathcal{G}^0)^* \otimes k\mathcal{G}^0) = (\omega\mathbb{1})^* \otimes \omega\mathbb{1}$
- $\pi((k\mathcal{G}^1)^* \otimes k\mathcal{G}^1) = (\omega M)^* \otimes \omega M$

and for the comodules

- $k\mathcal{G}^m$ pushes forward to $(\omega M)^{\otimes m}$

Next Step

Compute the kernel $I = \ker \pi$ and obtain a characterization

$$H[\mathcal{G}]/I \cong \mathbf{coend}(\mathcal{C}, \omega) \qquad \text{and} \qquad \mathcal{M}^{H[\mathcal{G}]/I} \simeq \mathcal{C}$$

$H[\mathcal{G}]$ Rules all Fusion Categories

Theorem

There is a surjection of Weak Bialgebras

$$\pi\colon H[\mathcal{G}] \to \mathbf{coend}(\mathcal{C}, \omega)$$

such that

- $\pi((k\mathcal{G}^0)^* \otimes k\mathcal{G}^0) = (\omega\mathbb{1})^* \otimes \omega\mathbb{1}$
- $\pi((k\mathcal{G}^1)^* \otimes k\mathcal{G}^1) = (\omega M)^* \otimes \omega M$

and for the comodules

- $k\mathcal{G}^m$ pushes forward to $(\omega M)^{\otimes m}$

Next Step

Compute the kernel $I = \ker \pi$ and obtain a characterization

$$H[\mathcal{G}]/I \cong \mathbf{coend}(\mathcal{C}, \omega) \quad \text{and} \quad \mathcal{M}^{H[\mathcal{G}]/I} \simeq \mathcal{C}$$

$H[\mathcal{G}]$ Rules all Fusion Categories

Theorem

There is a surjection of Weak Bialgebras

$$\pi\colon H[\mathcal{G}] \to \mathbf{coend}(\mathcal{C},\omega)$$

such that

- $\pi((k\mathcal{G}^0)^* \otimes k\mathcal{G}^0) = (\omega\mathbb{1})^* \otimes \omega\mathbb{1}$
- $\pi((k\mathcal{G}^1)^* \otimes k\mathcal{G}^1) = (\omega M)^* \otimes \omega M$

and for the comodules

- $k\mathcal{G}^m$ pushes forward to $(\omega M)^{\otimes m}$

Next Step

Compute the kernel $I = \ker \pi$ and obtain a characterization

$$H[\mathcal{G}]/I \cong \mathbf{coend}(\mathcal{C},\omega) \quad \text{and} \quad \mathcal{M}^{H[\mathcal{G}]/I} \simeq \mathcal{C}$$

$H[\mathcal{G}]$ Rules all Fusion Categories

Theorem

There is a surjection of Weak Bialgebras

$$\pi\colon H[\mathcal{G}] \to \mathbf{coend}(\mathcal{C}, \omega)$$

such that

- $\pi((k\mathcal{G}^0)^* \otimes k\mathcal{G}^0) = (\omega\mathbb{1})^* \otimes \omega\mathbb{1}$
- $\pi((k\mathcal{G}^1)^* \otimes k\mathcal{G}^1) = (\omega M)^* \otimes \omega M$

and for the comodules

- $k\mathcal{G}^m$ pushes forward to $(\omega M)^{\otimes m}$

Next Step

Compute the kernel $I = \ker \pi$ and obtain a characterization

$$H[\mathcal{G}]/I \cong \mathbf{coend}(\mathcal{C}, \omega) \qquad \text{and} \qquad \mathcal{M}^{H[\mathcal{G}]/I} \simeq \mathcal{C}$$

Modular categories associated with $U_q(\mathfrak{sl}_2)$

Example

I is the two-sided ideal generated by:

- quadratic relations $RTT - TTR$:

$$\sum_{(def)\in\mathcal{G}^2} \left(R_{(abc),(def)}[def|pqr]_2 - [abc|def]_2 R_{(def),(pqr)} \right)$$

- $\longrightarrow$ guarantee that $k\mathcal{G}^m$ decomposes as $(\omega M)^{\otimes m}$
- inhomogeneous relation (degree 0 and degree 2):

$$1 - \mathrm{qdet}$$

  (remove the group-like quantum determinant)
- $\longrightarrow$ guarantees that $\mathcal{M}^{H[\mathcal{G}]/I} \simeq \mathcal{C}$

Modular categories associated with $U_q(\mathfrak{sl}_2)$

Example

I is the two-sided ideal generated by:

- quadratic relations $RTT - TTR$:

$$\sum_{(def)\in\mathcal{G}^2}\left(R_{(abc),(def)}[def|pqr]_2 - [abc|def]_2 R_{(def),(pqr)}\right)$$

- $\longrightarrow$ guarantee that $k\mathcal{G}^m$ decomposes as $(\omega M)^{\otimes m}$
- inhomogeneous relation (degree 0 and degree 2):

$$1 - \mathrm{qdet}$$

  (remove the group-like quantum determinant)
- $\longrightarrow$ guarantees that $\mathcal{M}^{H[\mathcal{G}]/I} \simeq \mathcal{C}$

Modular categories associated with $U_q(\mathfrak{sl}_2)$

Example

I is the two-sided ideal generated by:

- quadratic relations $RTT - TTR$:

$$\sum_{(def)\in\mathcal{G}^2}\Big(R_{(abc),(def)}[def|pqr]_2 - [abc|def]_2 R_{(def),(pqr)}\Big)$$

- $\longrightarrow$ guarantee that $k\mathcal{G}^m$ decomposes as $(\omega M)^{\otimes m}$
- inhomogeneous relation (degree 0 and degree 2):

$$1 - \mathrm{qdet}$$

(remove the group-like quantum determinant)

- $\longrightarrow$ guarantees that $\mathcal{M}^{H[\mathcal{G}]/I} \simeq \mathcal{C}$

Modular categories associated with $U_q(\mathfrak{sl}_2)$

Example

I is the two-sided ideal generated by:

- quadratic relations $RTT - TTR$:

$$\sum_{(def)\in\mathcal{G}^2}\Big(R_{(abc),(def)}[def|pqr]_2 - [abc|def]_2 R_{(def),(pqr)}\Big)$$

- $\longrightarrow$ guarantee that $k\mathcal{G}^m$ decomposes as $(\omega M)^{\otimes m}$
- inhomogeneous relation (degree 0 and degree 2):

$$1 - \mathrm{qdet}$$

(remove the group-like quantum determinant)

- $\longrightarrow$ guarantees that $\mathcal{M}^{H[\mathcal{G}]/I} \simeq \mathcal{C}$

Modular categories associated with $U_q(\mathfrak{sl}_2)$

Example

I is the two-sided ideal generated by:

- quadratic relations $RTT - TTR$:

$$\sum_{(def)\in\mathcal{G}^2}\left(R_{(abc),(def)}[def|pqr]_2 - [abc|def]_2 R_{(def),(pqr)}\right)$$

- $\longrightarrow$ guarantee that $k\mathcal{G}^m$ decomposes as $(\omega M)^{\otimes m}$
- inhomogeneous relation (degree 0 and degree 2):

$$1 - \mathrm{qdet}$$

  (remove the group-like quantum determinant)
- $\longrightarrow$ guarantees that $\mathcal{M}^{H[\mathcal{G}]/I} \simeq \mathcal{C}$

Modular categories associated with $U_q(\mathfrak{sl}_2)$

Example (… continued)

Non-zero coefficients of $R_{(abc),(def)}$:

$$\begin{aligned}
R_{(j,j\pm1,j);(j,j\pm1,j)} &= \mp\zeta^{-1}\,\frac{q^{\pm(j+1)}}{[j+1]_q},\\
R_{(j,j-1,j);(j,j+1,j)} &= \zeta^{-1}\,\frac{[j]_q[j+2]_q}{{[j+1]_q}^2},\\
R_{(j,j+1,j);(j,j-1,j)} &= \zeta^{-1},\\
R_{(j,j\pm1,j\pm2);(j,j\pm1,j\pm2)} &= \zeta^{-1}q^{-1},
\end{aligned}$$

with $q=\zeta^2$, $[n]_q=(q^n-q^{-n})/(q-q^{-1})$.

Modular categories associated with $U_q(\mathfrak{sl}_2)$

Example (... continued)

Quantum determinant:

$$\begin{aligned}\mathrm{qdet} = \sum_{j,\ell=0}^{r-2}\Bigg(& \quad \frac{[\ell+1]_q}{[j+1]_q}\,[(j,j+1,j)|(\ell,\ell+1,\ell)]_2 \\ & + \quad \frac{[\ell]_q}{[j]_q}\,[(j,j-1,j)|(\ell,\ell-1,\ell)]_2 \\ & - \quad \frac{[\ell+1]_q}{[j]_q}\,[(j,j-1,j)|(\ell,\ell+1,\ell)]_2 \\ & - \quad \frac{[\ell]_q}{[j+1]_q}\,[(j,j+1,j)|(\ell,\ell-1,\ell)]_2\Bigg).\end{aligned}$$

Modular categories associated with $U_q(\mathfrak{sl}_2)$

Example (... continued)

The quotient $H = H[\mathcal{G}]/I$

- is a Weak Hopf Algebra
- is finite-dimensional, split cosemisimple
- is coribbon and weakly cofactorizable
- has commutative transversal base algebras
- is isomorphic to its dual

The category of finite-dimensional comodules $\mathcal{M}^H$ is equivalent to the modular category associated with $U_q(\mathfrak{sl}_2)$

Modular categories associated with $U_q(\mathfrak{sl}_2)$

Example (… continued)

The quotient $H = H[\mathcal{G}]/I$

- is a Weak Hopf Algebra
- is finite-dimensional, split cosemisimple
- is coribbon and weakly cofactorizable
- has commutative transversal base algebras
- is isomorphic to its dual

The category of finite-dimensional comodules $\mathcal{M}^H$ is equivalent to the modular category associated with $U_q(\mathfrak{sl}_2)$

Modular categories associated with $U_q(\mathfrak{sl}_2)$

Example (... continued)

The quotient $H = H[\mathcal{G}]/I$

- is a Weak Hopf Algebra
- is finite-dimensional, split cosemisimple
- is coribbon and weakly cofactorizable
- has commutative transversal base algebras
- is isomorphic to its dual

The category of finite-dimensional comodules $\mathcal{M}^H$ is equivalent to the modular category associated with $U_q(\mathfrak{sl}_2)$

Modular categories associated with $U_q(\mathfrak{sl}_2)$

Example (... continued)

The quotient $H = H[\mathcal{G}]/I$

- is a Weak Hopf Algebra
- is finite-dimensional, split cosemisimple
- is coribbon and weakly cofactorizable
- has commutative transversal base algebras
- is isomorphic to its dual

The category of finite-dimensional comodules $\mathcal{M}^H$ is equivalent to the modular category associated with $U_q(\mathfrak{sl}_2)$

Modular categories associated with $U_q(\mathfrak{sl}_2)$

Example (. . . continued)

The quotient $H = H[\mathcal{G}]/I$

- is a Weak Hopf Algebra
- is finite-dimensional, split cosemisimple
- is coribbon and weakly cofactorizable
- has commutative transversal base algebras
- is isomorphic to its dual

The category of finite-dimensional comodules $\mathcal{M}^H$ is equivalent to the modular category associated with $U_q(\mathfrak{sl}_2)$

Modular categories associated with $U_q(\mathfrak{sl}_2)$

Example (... continued)

The quotient $H = H[\mathcal{G}]/I$

- is a Weak Hopf Algebra
- is finite-dimensional, split cosemisimple
- is coribbon and weakly cofactorizable
- has commutative transversal base algebras
- is isomorphic to its dual

The category of finite-dimensional comodules $\mathcal{M}^H$ is equivalent to the modular category associated with $U_q(\mathfrak{sl}_2)$

Modular categories associated with $U_q(\mathfrak{sl}_2)$

Example (... continued)

The quotient $H = H[\mathcal{G}]/I$

- is a Weak Hopf Algebra
- is finite-dimensional, split cosemisimple
- is coribbon and weakly cofactorizable
- has commutative transversal base algebras
- is isomorphic to its dual

The category of finite-dimensional comodules $\mathcal{M}^H$ is equivalent to the modular category associated with $U_q(\mathfrak{sl}_2)$

Summary

Theorem

Every multi-fusion category $\mathcal{C}$ is of the form $\mathcal{C} \simeq \mathcal{M}^{H[\mathcal{G}]/I}$ where $\mathcal{G}$ is its dimension graph with respect to some fusion generator $M \in |\mathcal{C}|$ and the two-sided ideal I is generated by two types of relations

- type 1 (homogeneous): implement $\mathrm{End}((\omega M)^{\otimes m})$ for all $m = 0, 1, 2, \ldots$
- type 2 (inhomogeneous): remove suitable group-likes and make $H[\mathcal{G}]/I$ a finite-dimensional Weak Hopf Algebra

You can make the tensor product of every fusion category trivial provided that you understand the ideal I above.

Summary

Theorem

Every multi-fusion category $\mathcal{C}$ is of the form $\mathcal{C} \simeq \mathcal{M}^{H[\mathcal{G}]/I}$ where $\mathcal{G}$ is its dimension graph with respect to some fusion generator $M \in |\mathcal{C}|$ and the two-sided ideal I is generated by two types of relations

- type 1 (homogeneous): implement $\mathrm{End}((\omega M)^{\otimes m})$ for all $m = 0, 1, 2, \ldots$
- type 2 (inhomogeneous): remove suitable group-likes and make $H[\mathcal{G}]/I$ a finite-dimensional Weak Hopf Algebra

You can make the tensor product of every fusion category trivial provided that you understand the ideal I above.

Summary

Theorem

Every multi-fusion category $\mathcal{C}$ is of the form $\mathcal{C} \simeq \mathcal{M}^{H[\mathcal{G}]/I}$ where $\mathcal{G}$ is its dimension graph with respect to some fusion generator $M \in |\mathcal{C}|$ and the two-sided ideal I is generated by two types of relations

- type 1 (homogeneous): implement $\mathrm{End}((\omega M)^{\otimes m})$ for all $m = 0, 1, 2, \ldots$
- type 2 (inhomogeneous): remove suitable group-likes and make $H[\mathcal{G}]/I$ a finite-dimensional Weak Hopf Algebra

You can make the tensor product of every fusion category trivial provided that you understand the ideal I above.

Summary

Theorem

Every multi-fusion category $\mathcal{C}$ is of the form $\mathcal{C} \simeq \mathcal{M}^{H[\mathcal{G}]/I}$ where $\mathcal{G}$ is its dimension graph with respect to some fusion generator $M \in |\mathcal{C}|$ and the two-sided ideal I is generated by two types of relations

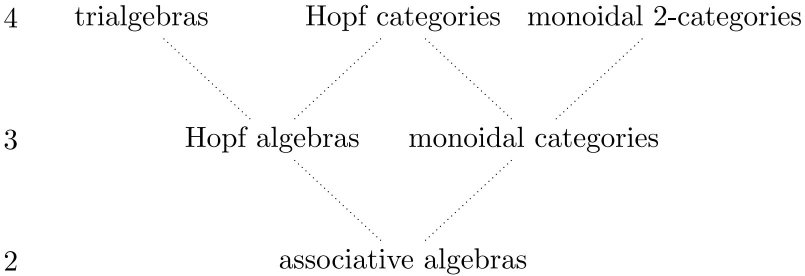
- type 1 (homogeneous): implement $\mathrm{End}((\omega M)^{\otimes m})$ for all $m = 0, 1, 2, \ldots$
- type 2 (inhomogeneous): remove suitable group-likes and make $H[\mathcal{G}]/I$ a finite-dimensional Weak Hopf Algebra

You can make the tensor product of every fusion category trivial provided that you understand the ideal I above.

What next?

Categorification [Crane–Frenkel]

Algebraic structures used for combinatorial invariants of low-dimensional (smooth) manifolds:



What next?

Categorification [Crane–Frenkel]

Algebraic structures used for combinatorial invariants of low-dimensional (smooth) manifolds:

4 trialgebras Hopf categories monoidal 2-categories

3 Hopf algebras monoidal categories

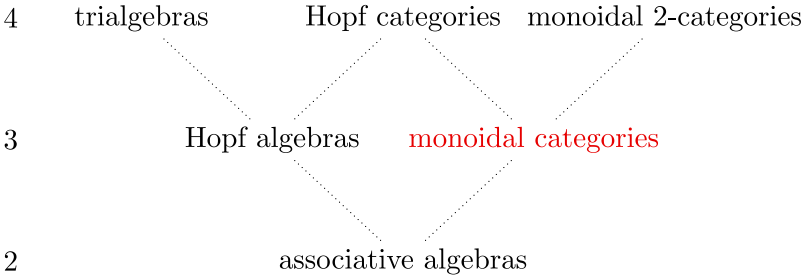
2 associative algebras

Finite-dimensional separable algebra [Fukuma–Hosono–Kawai]

What next?

Categorification [Crane–Frenkel]

Algebraic structures used for combinatorial invariants of low-dimensional (smooth) manifolds:

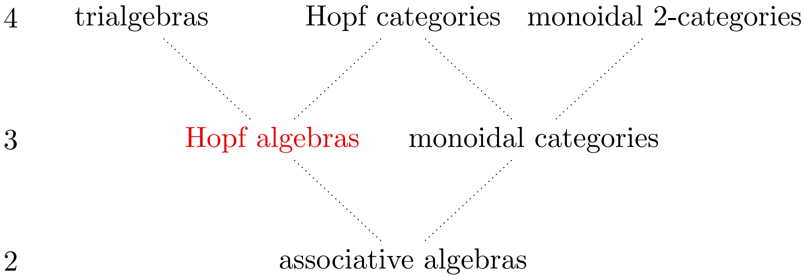


Spherical fusion category [Turaev–Viro, Barrett–Westbury]

What next?

Categorification [Crane–Frenkel]

Algebraic structures used for combinatorial invariants of low-dimensional (smooth) manifolds:

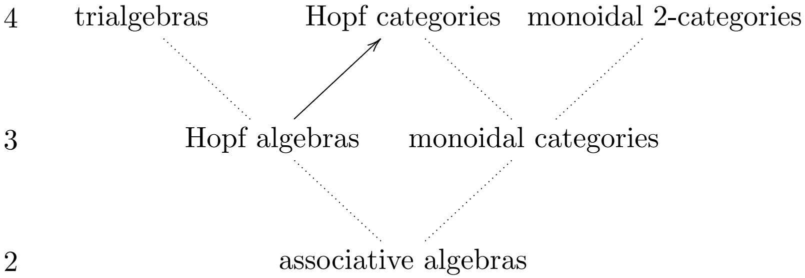


Finite-dimensional involutory Hopf algebra [Kuperberg]

What next?

Categorification [Crane–Frenkel]

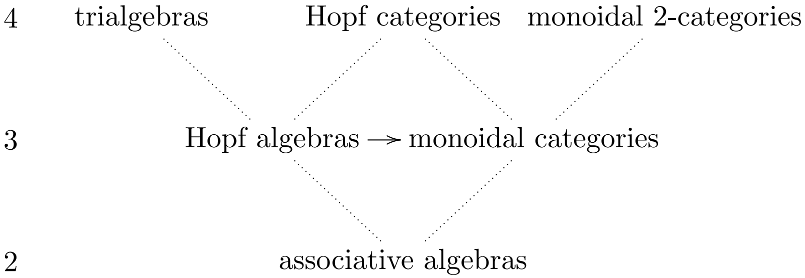
Algebraic structures used for combinatorial invariants of low-dimensional (smooth) manifolds:



What next?

Categorification [Crane–Frenkel]

Algebraic structures used for combinatorial invariants of low-dimensional (smooth) manifolds:

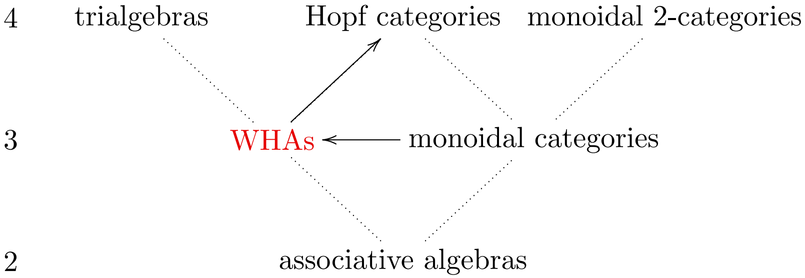


$H \mapsto {}_H\mathcal{M}$ [Barrett–Westbury]

What next?

Categorification [Crane–Frenkel]

Algebraic structures used for combinatorial invariants of low-dimensional (smooth) manifolds:



References

References

- H. Pfeiffer, Tannaka-Krein reconstruction and a characterization of modular tensor categories *J. Alg.* **321**, No. 12 (2009) 3714-3763, `arXiv:0711.1402`
- H. Pfeiffer, Finitely semisimple spherical categories and modular categories are self-dual *Adv. Math.* **221**, No. 5 (2009) 1608-1652, `arXiv:0806.2903`
- H. Pfeiffer, Fusion categories in terms of graphs and relations `arXiv:0912.0342`