Combinatorial characterization of fusion categories

(Erlangen, Nov 1, 2011)

by Hendryk Pfeiffer, UBC

[1] J Alg 321 (2009) 3714

[2] Q Top 2 (2011) 339

1. Introduction:

$U_q(sl_2)$: $q \in \mathbb{C}$, $q^p \neq 1$ $\forall p \in \mathbb{N}$

$$U_q(sl_2) = \mathbb{C}\{E, F, K, K^{-1}\}/(\text{relations})$$

relations:

$$KK^{-1} = 1 = K^{-1}K$$

$$KEK = q^2 E$$

$$KFK = q^{-2} F$$

$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}$$

structure:

$$\Delta E = 1 \otimes E + E \otimes K \qquad \varepsilon E = 0 \qquad SE = -EK^{-1}$$

$$\Delta F = K^{-1} \otimes F + F \otimes 1 \qquad \varepsilon F = 0 \qquad SF = -KF$$

$$\Delta K = K \otimes K \qquad \varepsilon K = 1 \qquad SK = K^{-1}$$

$$\Delta K^{-1} = K^{-1} \otimes K^{-1} \qquad \varepsilon K^{-1} = 1 \qquad SK^{-1} = K$$

$U_q(sl_2)\mathcal{M}$: $V_{\varepsilon,n}$ $\quad \varepsilon \in \{-1, 1\}$, $n = 0, 1, 2, \ldots$ $\quad \dim V_{\varepsilon,n} = n+1$

basis $\{v_0, \ldots, v_n\}$:

$$K v_p = \varepsilon q^{n-2p} v_p$$

$$E v_p = \varepsilon [n-p+1] v_{p-1}$$

$$F v_p = [p] v_{p+1}$$

Other point of view [Faddeev, Reshetikhin, Takhtajan 1990]

$$SL_q(2) = \mathbb{C}\{t_{ij} \mid 1 \le i,j \le 2\}/(\text{relations})$$

relations: a) quadratic "RTT = TTR"

$$0 = \sum_{k,\ell} \left(R^{k\ell}_{ij} T^p_k T^q_\ell - T^k_i T^\ell_j R^{pq}_{k\ell} \right) \qquad R = \begin{pmatrix} q & & & \\ & 0 & 1 & \\ & 1 & q-q^{-1} & \\ & & & q \end{pmatrix}$$

b) inhomogeneous "qdet = 1"

$$0 = \underbrace{t_{22}t_{11} - q t_{12} t_{21}}_{\text{qdet}} - 1$$

Relationship: $\quad {}_{U_q(sl_2)}\mathcal{M}^{\varepsilon=1} \simeq \mathcal{M}^{SL_q(2)}$

Q: $\quad \mathcal{C}$ autonomous monoidal $\overset{?}{\Longrightarrow}$ $\mathcal{C} \simeq \mathcal{M}^H$

$\quad \mathcal{C} = {}_{U_q(g)}\mathcal{M}$ [Müller 2001]

here: $\quad \mathcal{C}$ = (multi-) fusion

2. Fusion Categories

Def: $\mathcal{C}$ multi-fusion:

- autonomous monoidal
- additive (has finite biproducts '$\oplus$')
- k-linear, $k = \mathrm{End}(\mathbb{1})$
- finitely split semisimple (split: $\mathrm{End}(X) \cong k$ for X simple)
- $\mathcal{C}$ essentially small; k field; $\mathrm{Hom}(X,Y)$ f.d. $\forall X,Y \in |\mathcal{C}|$.
- $\Rightarrow \mathcal{C}$ abelian

$\mathcal{C}$ fusion:

- $\mathbb{1}$ simple

Example: 1) G finite group: ${}_{\mathbb{C}[G]}\mathcal{M}$ is fusion

2) $\mathcal{C}_p$ = SL_2 fusion category

$p = 3, 4, 5, \ldots$

$\zeta^{4p} = 1$ (primitive $4p$-th root of unity)

$\mathcal{C}_p$ = quotient of $\hat{U}^{fin}_{\zeta}(sl_2)$ tilting modules modulo negligible morphisms

representatives of simple objects: $(V_j)_{j \in I}$, $j \in \{0, \ldots, p-2\}$

Prop: H Hopf algebra over k
The forgetful functor $U: {}_H\mathcal{M} \longrightarrow Vect_k$ is k-linear, faithful, exact and strong monoidal, i.e. $U(X \otimes Y) \cong UX \otimes_k UY$, $U\mathbb{1} \cong k$.

Thm: [Deligne, ..., Etingof-Ostrik]
$\mathcal{C}_p$ does not admit any $\mathbb{C}$-linear, faithful, exact and strong monoidal functor $U: \mathcal{C}_p \to Vect_{\mathbb{C}}$.
($\to$ Frobenius-Perron dimension)

3. Reconstruction:

Thm: [Hayashi, Hai]
$\mathcal{C}$ multi-fusion, $\hat{V} = \bigoplus_{j \in I} V_j$ small projective generator

The long canonical functor

$$\omega: \mathcal{C} \longrightarrow Vect_k, \quad X \longmapsto Hom(\hat{V}, \hat{V} \otimes X)$$

is k-linear, faithful and exact.

Thm: (Classical)

$$H = \mathrm{Coend}(\mathcal{C}, \omega) = \bigoplus_{j \in I} (\omega V_j)^* \underset{\substack{k \\ \Psi \\ [v \mid w]_{V_j}}}{\otimes} \omega V_j$$

- coassociative counital coalgebra over k
- finite-dimensional split cosemisimple
- $\mathcal{C} \simeq \mathcal{M}^H$ as k-linear additive categories

Q: Monoidal structure?

Thm: The long canonical functor $\omega : \mathcal{C} \to \mathrm{Vect}_k$
- is lax-, oplax-monoidal
- has separable Frobenius structure
- $R = \mathrm{End}(\hat{V})$ is index-one Frobenius, commutative

$H = \mathrm{coend}(\mathcal{C}, \omega)$
- is a WHA
- $H_t \cong R$, $H_s \cong R$ commutative base algebras
- $\mathcal{C} \simeq \mathcal{M}^H$ as k-linear additive monoidal categories

Remark:

[Hayashi, Hai, Ostrik] "short canonical functor" strong monoidal

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[dr, "\omega"] & {}_R\mathcal{M}_R \arrow[d] \\ & \mathrm{Vect}_k \end{tikzcd}$$

„long canonical"

Thm:

$$\begin{tikzcd} \mathrm{Cat}\downarrow\mathrm{Vect}_k^{\otimes *} \arrow[rr, bend left, "\mathrm{coend}(-)"] \arrow[rr, phantom, "\perp"] & & \mathrm{WHA}_k \arrow[ll, bend left, "\mathcal{M}^-"] \end{tikzcd}$$

$\varepsilon : 1_{\mathrm{WHA}} \Rightarrow \mathrm{Coend}(\mathcal{M}^-)$ isomorphism

$\eta : \mathcal{M}^{\mathrm{Coend}(-)} \Rightarrow 1_{\mathrm{Cat}\downarrow\mathrm{Vect}_k^{\otimes *}}$ isomorphism provided

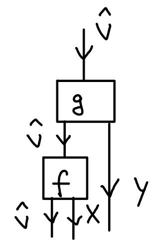
$\mathcal{C}$ is k-linear, abelian

ω is k-linear, faithful, exact

4. Characterization of $H = \mathrm{Coend}(\mathcal{C}, \omega)$

lax monoidal structure:

$$\begin{array}{ccc} \omega_{X,Y} : \omega X \otimes \omega Y & \longrightarrow & \omega(X \otimes Y) \\ \mathrm{Hom}(\hat{V}, \hat{V}\otimes X)\ \ \mathrm{Hom}(\hat{V}, \hat{V}\otimes Y) & & \mathrm{Hom}(\hat{V}, \hat{V}\otimes(X\otimes Y)) \\ f \otimes g & \longmapsto & \alpha_{\hat{V},X,Y} \circ (f \otimes \mathrm{id}_Y) \circ g \end{array}$$



If $f : V_\ell \longrightarrow V_m \otimes X$, $g : V_i \longrightarrow V_j \otimes Y$, then $\omega_{XY}(f \otimes g) = 0$ unless $\ell = j$.

$\uparrow$ source $\uparrow$ target $\uparrow$ composition

Def: $\mathcal{C}$ multi-fusion, $M \in |\mathcal{C}|$ monoidal generator

i.e. $\forall j \in I\ \exists m \in \mathbb{N}_0 : M^{\otimes m} \cong V_j \oplus (\text{something})$

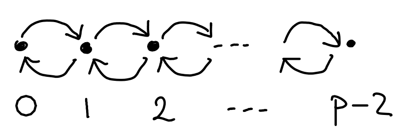
The <u>dimension graph</u> $\mathcal{G}$ of $(\mathcal{C}, M)$ has

vertices: $\mathcal{G}^0 := I$

edges $j \to \ell$: $\mathcal{G}^1_{j\ell} = $ basis of $\omega M = \mathrm{Hom}(V_j, V_\ell \otimes M)$

notation: $\mathcal{G}^m = $ paths of length $m \geq 0$ in $\mathcal{G}$

<u>Example:</u> $\mathcal{C}_p$ with $M = V_1$: $\mathcal{G} =$



<u>Thm:</u> The path algebra $H[\mathcal{G}] := k(\mathcal{G}\times\mathcal{G})$ is a graded algebra

$$H[\mathcal{G}] = \bigoplus_{m\geq 0} \overbrace{(k\mathcal{G}^m)^* \otimes (k\mathcal{G}^m)}^{H[\mathcal{G}]_m} \ni [p|q]_m$$

and a split cosemisimple WBA with operations

$$\mu([p|q]_m \otimes [r|s]_n) = \delta_{\sigma(p),\tau(r)}\,\delta_{\sigma(q),\tau(s)}\,[pr, qs]_{m+n}$$

$$\eta(1) = \sum_{i,j\in I} [i|j]$$

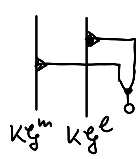
$$\Delta([p|q]_m) = \sum_{r\in\mathcal{G}^m} [p|r]_m \otimes [r|q]_m$$

$$\varepsilon([p|q]_m) = \delta_{pq}$$

generated by $H[\mathcal{G}]_0 \cup H[\mathcal{G}]_1$

simple comodules: $k\mathcal{G}^m$, $m\geq 0$: $\qquad \beta(p) = \sum_{q\in\mathcal{G}^m} q\otimes [q|p]_m$

monoidal structure: $k\mathcal{G}^m \otimes k\mathcal{G}^e \cong k\mathcal{G}^{m+e} \subseteq k\mathcal{G}^m \underset{k}{\otimes} k\mathcal{G}^e$



$$(p\otimes q) = \begin{cases} pq & \text{if } \sigma(p)=\tau(q) \\ 0 & \text{else} \end{cases}$$

<u>Thm:</u> $\mathcal{C}$ multi-fusion, $H = \mathrm{coend}(\mathcal{C},\omega)$

1) There is a unique homomorphism of algebras $\pi: H[\mathcal{G}] \to H$ such that

$$\pi([i|j]_0) = [\lambda_i|\lambda_j]_{\mathbb{1}}$$

$$\pi([p|q]_1) = [p|q]_M$$

2) π is a surjection of WBAs.

→ The path algebra $k(\mathcal{G}\times\mathcal{G})$ controls the $\otimes$-product in fusion categories

→ If we can compute $\ker\pi$, we have won!

<u>Def:</u> $\mathcal{C}$ multi-fusion, M monoidal generator, $\mathcal{G}$ dimension graph
$E^{(n)} \subseteq \mathrm{End}(M^{\otimes n})$ generating set

Coeff. of $\omega(f^{(n)})$, $f^{(n)} \in E^{(n)}$

$$H[\mathcal{G},E] = H[\mathcal{G}] \Big/ \Big(\sum_{p_j \in \mathcal{G}^1} [r_1|p_1]_1 \cdots [r_n|p_n]_1 \, f^{(n)}_{p_1 \cdots p_n, q_1 \cdots q_n} - f^{(n)}_{r_1 \cdots r_n, p_1 \cdots p_n} [p_1|q_1]_1 \cdots [p_n|q_n]_n \Big)$$

<u>Example:</u> $\mathcal{C}_p$:
$$\sum_{p_1,p_2 \in \mathcal{G}^1} [r_1|p_1]_1 [r_2|p_2]_1 R_{p_1p_2,q_1q_2} - R_{r_1r_2,p_1p_2} [p_1|q_1]_1 [p_2|q_2]_1 \tag{1}$$

with
$$R_{(jj-1j)(jj+1j)} = \varsigma^{-1} \frac{[j]\ [j+2]}{[j+1]^2}, \quad R_{(jj+1j)(jj+1j)} = -\varsigma^{-1} \frac{\varsigma^{2j+2}}{[j+1]}$$

$$R_{(jj-1j)(jj-1j)} = \varsigma^{-1} \frac{\varsigma^{-2j-2}}{[j+1]}, \quad R_{(jj+1j)(jj-1j)} = \varsigma^{-1}$$

$$R_{(jj\pm1j\pm2)(jj\pm1j\pm2)} = \varsigma^{-3}, \quad R_{\text{other}} = 0$$

$$[n] = \frac{\varsigma^{2n} - \varsigma^{-2n}}{\varsigma^2 - \varsigma^{-2}}, \quad n \in \mathbb{Z}.$$

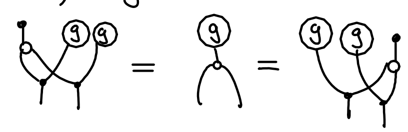
<u>Thm:</u>
$$\begin{tikzcd} H[\mathcal{G}] \arrow[r] \arrow[dr, "\pi"'] & H[\mathcal{G},E] \arrow[d, "\overline{\pi}"] \\ & H = \mathrm{coend}(\mathcal{C},\omega) \end{tikzcd}$$
factors through

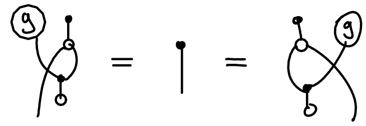
$f : \omega(M^{\otimes m}) \to \omega(M^{\otimes m})$ is $H[\mathcal{G},E]$-colinear iff H-colinear

<u>Example:</u> $\mathcal{C}_p$, p large:

m	$\omega(M^{\otimes m}) \in \lvert\mathcal{M}^H\rvert$	$k\mathcal{G}^m \in \lvert\mathcal{M}^{H[\mathcal{G},E]}\rvert$
0	V_0	V_0
1	V_1	V_1
2	$V_0 \oplus V_2$	$V_0' \oplus V_2$
3	$2V_1 \oplus V_3$	$2V_1' \oplus V_3$
4	$2V_0 \oplus 3V_2 \oplus V_4$	$2V_0'' \oplus 3V_2' \oplus V_4$

<u>Def:</u> H WBA, $g \in H$ is called <u>group-like</u> if

a) 

b) 

<u>Thm:</u> $\operatorname{Ker} \pi = (1-g \mid g \text{ group-like and } \pi(g)=1)$

<u>Example:</u> $\mathcal{C}_p \simeq \mathcal{M}^H$, $H = H[\mathcal{G}]/(\text{eq.}(1), \text{eq.}(2))$

$$1 - \sum_{j,\ell \in \mathcal{G}^0} \alpha_j \alpha_\ell \Big(\frac{[\ell+1]}{[j+1]} [j\,j{+}1\,j \mid \ell\,\ell{+}1\,\ell]_2 - \frac{[\ell+1]}{[j]} [j\,j{-}1\,j \mid \ell\,\ell{+}1\,\ell]_2 - \frac{[\ell]}{[j+1]} [j\,j{+}1\,j \mid \ell\,\ell{-}1\,\ell]_2 + \frac{[\ell]}{[j]} [j\,j{-}1\,j \mid \ell\,\ell{-}1\,\ell]_2 \Big)$$

$\alpha_0 = \alpha_{p-2} = 1$

$\alpha_j = \frac{1}{\sqrt{2}}$ if $j = 1, \dots, p-3$.

<u>Thm:</u> Every multi-fusion category $\mathcal{C}$ is equivalent to $\mathcal{M}^H$ for a WHA $H = H[\mathcal{G}]/(\text{relations A}, \text{relations B})$

$\mathcal{G}$: dimension graph of $\mathcal{C}$ w.r.t. a monoidal generator M

relations A: generalized "RTT=TTR" enforce $\operatorname{End}(\omega(M^{\otimes m}))$, eq. (1)

relations B: generalized "qdet − 1" compare different m, eq. (2).