Fusion categories in terms of graphs and relations

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Outline

1 Fusion categories

2 Weak Hopf Algebras

3 Tannaka reconstruction

4 Characterization of fusion categories

1 Fusion categories

Def: Multi-fusion category: essentially small finitely split semisimple k-linear additive rigid monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ such that

a) k field

b) $\mathrm{Hom}(X,Y)$ finite-dimensional $\forall X, Y \in |\mathcal{C}|$

(split semisimple: $\mathrm{End}(X) \cong k$ for simple X)

Fusion category: c) $\mathbb{1}$ simple

Example: 1) G finite group, $\mathbb{C}[G]$-mod is fusion category

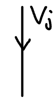
2) $\mathcal{C}_{U_q(sl_2),p}$ modular category associated with $U_q(sl_2)$, $p = 3,4,5,\dots$ (yesterday's talk)

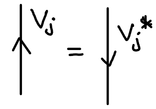
$k \in \mathbb{C}$

q primitive $4p$-th root of unity, $t = q^2$

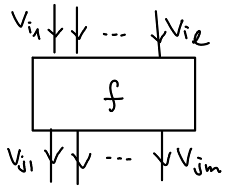
$(V_j)_{j \in I}$ represent isomorphism classes of simple objects, $I = \{0, 1, \dots, p-2\}$

Remark: Although $\mathcal{C}_{U_q(sl_2),p}$ is semisimple, it is not boring.
What is interesting and difficult is the tensor product.

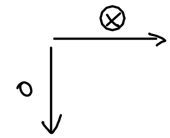
diagrams: $id_{V_j} =$ 

, $id_{V_j^*} =$ 

$id_{\mathbb{1}} = id_{V_0}$ invisible



$= f : V_{i_1} \otimes \cdots \otimes V_{i_l} \to V_{j_1} \otimes \cdots \otimes V_{j_m}$



$ev_{V_j} =$ 

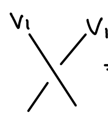
, $coev_{V_j} =$ 

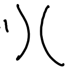
in general: $(V_j)^* \cong V_{j^*}$ for some $j^* \in I$

here: $j^* = j$ <u>and</u> $(V_j)^* = V_j$ $\longrightarrow$ omit all arrows

$q\dim V_j =$ 

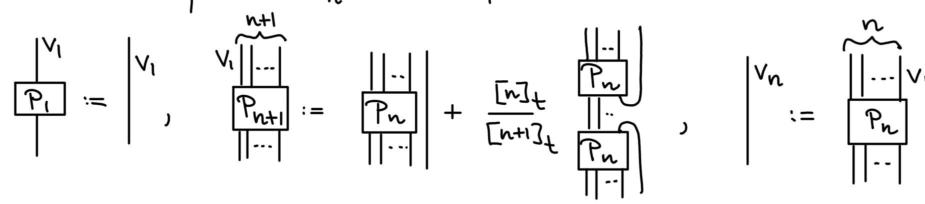
$= (-1)^n [n+1]_t$, $[k]_t = \frac{t^k - t^{-k}}{t - t^{-1}}$ $(t = q^2)$

ribbon structure: 

$= A$  $+ A^{-1}$ 

twist $=$  (all diagrams in blackboard framing)

Jones-Wenzl idempotents $P_n : V_1^{\otimes n} \to V_1^{\otimes n}$



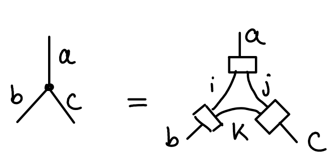
<u>Def:</u> $(a,b,c) \in I^3$ is <u>admissible</u> if

(i) $a+b+c \equiv 0 \mod 2$

(ii) $a+b-c \geq 0$, $b+c-a \geq 0$, $c+a-b \geq 0$

(iii) $a+b+c \leq 2r-4$

$$\mathrm{Hom}(V_a, V_b \otimes V_c) \cong \begin{cases} k & \text{if } (a,b,c) \text{ admissible} \\ 0 & \text{else} \end{cases}$$



$i = \frac{a+b-c}{2}$, $j = \frac{c+a-b}{2}$, $k = \frac{b+c-a}{2}$

Q: Is $\mathcal{C}_{U_q(sl_2),p} \simeq_{\otimes}$ H-mod for some algebra H with extra structure?

Prop: If H is a Hopf-algebra over k, then the forgetful functor

$U: H\text{-mod} \to Vect_k$

is k-linear, faithful, exact and strong monoidal,

i.e. in particular $U(X \otimes Y) \cong UX \otimes_k UY$, $U\mathbb{1} \cong k$.

Thm: $\mathcal{C}$ fusion category, $k = \mathbb{C}$

1) [Deligne, ..., Etingof-Ostrik] If $\mathcal{C}$ has a simple object of non-integer Frobenius-Perron dimension, then there exists no k-linear, faithful, exact strong monoidal functor $\mathcal{C} \to Vect_k$
2) $\mathcal{C}_{U_q(sl_2),p}$ has simple objects of non-integer FP dimension
3) $\mathcal{C}_{U_q(sl_2),p} \not\simeq_{\otimes}$ H-mod for any Hopf algebra H.

Q: What then is $\mathcal{C}_{U_q(sl_2)}$?

2 Weak Hopf Algebras

Def: [Böhm-Nill-Szlachányi]

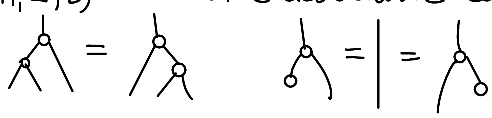
A Weak Bialgebra $(H, \mu, \eta, \Delta, \varepsilon)$ consists of

1) (H, μ, η) unital associative algebra

 $\mu : H \otimes H \to H$

 $\eta : k \to H$

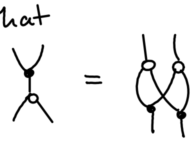
2) (H, Δ, ε) counital coassociative coalgebra

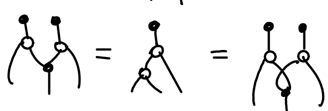
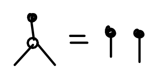


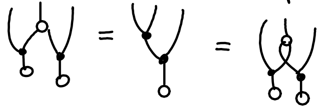
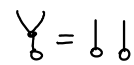
 $\Delta : H \to H \otimes H$

 $\varepsilon : H \to k$

such that

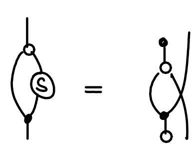
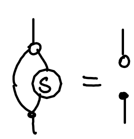
a)  (as in a Hopf algebra)

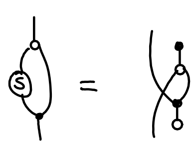
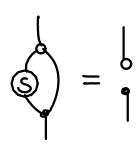
b)  (replaces )

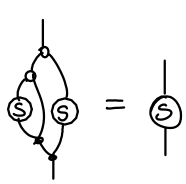
c)  (replaces )

(there is no analogue of  $= 1$)

A Weak Hopf Algebra $(H, \mu, \eta, \Delta, \varepsilon, S)$ is a WBA $(H, \mu, \eta, \Delta, \varepsilon)$ with a linear map $S : H \to H$ (antipode) such that

d)  $=: \varepsilon_s$ (replaces )

e)  $=: \varepsilon_t$ (replaces )

f)  (new)

Example: $G = (G^0, G^1, \sigma, \tau, \circ, id)$ finite groupoid, k field.

The groupoid algebra kG^1 is a WHA

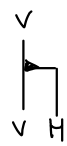
$$\mu(g \otimes h) = \begin{cases} g \circ h & \text{if } \sigma(g) = \tau(h) \\ 0 & \text{else} \end{cases} \qquad \forall g, h \in G^1$$

$$\eta(1) = \sum_{j \in G^0} id_j$$

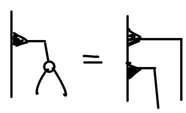
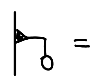
$$\Delta(g) = g \otimes g$$

$$\varepsilon(g) = 1$$

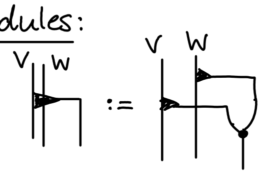
$$S(g) = g^{-1}$$

Comodule: $\beta_V =$  $: V \longrightarrow V \otimes H$

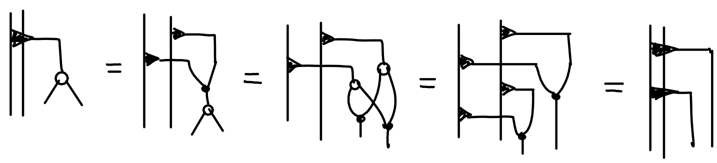
Conditions:

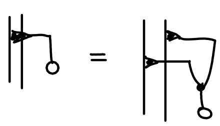
(1)  and (2) 

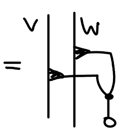
tensor product of comodules:

define:  (as with a Hopf algebra)

check conditions:

(1)  ✓

(2)  equal? $\longleftrightarrow$ 

But: $P_{V,W} =$  is an idempotent and we can define

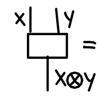
Def: $V \hat{\otimes} W := \operatorname{im} P_{V,W} \subseteq V \otimes_k W$ (truncated tensor product)

→ now look at the forgetful functor $\mathcal{M}^H \to \mathrm{Vect}_k$.

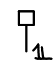
<u>Def:</u> [Szlachányi]

$\mathcal{C}, \mathcal{C}'$ monoidal categories. A functor with <u>separable Frobenius structure</u> $(F, F_0, F_{-,-}, F^0, F^{-,-}) : \mathcal{C} \to \mathcal{C}'$ consists of

1) $(F, F_0, F_{-,-}) : \mathcal{C} \to \mathcal{C}'$ lax monoidal

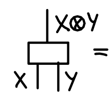


$= F_{X,Y} : FX \otimes' FY \to F(X \otimes Y)$



$= F_0 : \mathbb{1}' \to F\mathbb{1}$

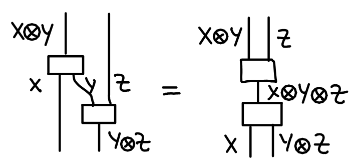
2) $(F, F^0, F^{-,-}) : \mathcal{C} \to \mathcal{D}$ oplax monoidal

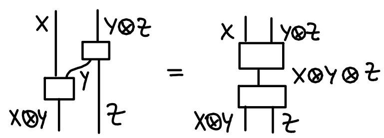


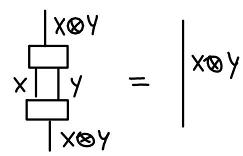
$= F^{X,Y} : F(X \otimes Y) \to FX \otimes' FY$

$= F^0 : F\mathbb{1} \to \mathbb{1}'$

such that

a) 

b) 

c) 

<u>Prop:</u> Let H be a WBA. The category of f.d. comodules $\mathcal{M}^H$ is monoidal and the forgetful functor

$U : \mathcal{M}^H \to \mathrm{Vect}_k$

is k-linear, faithful, exact with a separable Frobenius structure.

3 Tannaka reconstruction

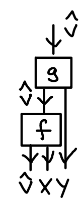
Thm: $\mathcal{C}$ multi-fusion category. Then the <u>long canonical functor</u>

$$\omega: \mathcal{C} \longrightarrow \mathrm{Vect}_k, \quad X \longmapsto \mathrm{Hom}(\hat{V}, \hat{V}\otimes X), \quad f \longmapsto (\mathrm{id}_{\hat{V}} \otimes f)\circ -$$

$$\hat{V} = \bigoplus_{j\in I} V_j$$

is k-linear, faithful, exact with a separable Frobenius structure.
(uses finite split semisimplicity)

lax monoidal structure: $\omega_{X,Y}: \omega X \otimes \omega Y \longrightarrow \omega(X\otimes Y)$
$$f\otimes g \longmapsto \alpha_{\hat{V},X,Y}\circ(f\otimes \mathrm{id}_Y)\circ g$$



$$\omega_0: k \longrightarrow \omega \mathbb{1}, \quad 1 \longmapsto \rho_{\hat{V}}^{-1}$$

<u>Remark:</u> a) $R = \mathrm{End}(\hat{V}) \cong k^{|I|}$ is index-one Frobenius $(R, \circ, \mathrm{id}_{\hat{V}}, \Delta_R, \varepsilon_R)$ with

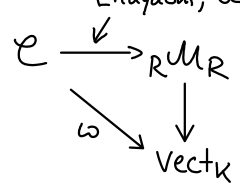
$$\Delta_R(\lambda_j) = \lambda_j \otimes \lambda_j$$
$$\varepsilon_R(\lambda_j) = 1 \qquad \lambda_j = \mathrm{id}_{V_j} \in R$$

(<u>index-one</u>: $\Delta_R(\mathrm{id}_{\hat{V}})$ is separability idempotent)

b) $g_X: \mathrm{Hom}(\hat{V}\otimes X, \hat{V}) \otimes \mathrm{Hom}(\hat{V}, \hat{V}\otimes X) \longrightarrow k$
$$\vartheta \otimes v \longmapsto \varepsilon_R(\vartheta \circ v)$$
are non-degenerate $\forall X \in |\mathcal{C}|$.

$\rightarrow \quad (\omega X)^* = \mathrm{Hom}(\hat{V}\otimes X, \hat{V}), \quad (\omega f)^* \vartheta = \vartheta \circ (\mathrm{id}_{\hat{V}} \otimes f)$ for $\vartheta \in (\omega Y)^*, f: X \to Y$

c) [Hayashi, Ostrik] "short canonical functor"
(strong monoidal, i.e. $\omega(X\otimes Y) \cong \omega X \otimes_R \omega Y$)



Thm: $\mathcal{C}$ multi-fusion category.

1) $H = \mathrm{Coend}(\mathcal{C}, \omega)$ is a finite-dimensional split cosemisimple WHA.

in general: $$\mathrm{coend}(\mathcal{C},\omega) = \Big(\coprod_{X \in |\mathcal{C}|} (\omega X)^* \otimes \omega X\Big) \Big/ \Big\langle [(\omega f)^* \vartheta | v]_X - [\vartheta | (\omega f) v]_Y \;\Big|\; \begin{matrix} f: X \to Y \\ v \in \omega X \\ \vartheta \in (\omega Y)^* \end{matrix} \Big\rangle$$
with $[\vartheta | v]_X \in (\omega X)^* \otimes \omega X$

here: $$H = \bigoplus_{j \in I} (\omega V_j)^* \otimes \omega V_j$$

$$\eta(1) = [\rho_{\hat{v}} | \rho_{\hat{v}}^{-1}]_{\mathbb{1}}$$
$$\mu([\vartheta|v]_X \otimes [\zeta|w]_Y) = [\zeta \circ (\vartheta \otimes id_Y) \circ \alpha^{-1}_{\hat{v}XY} \,|\, \alpha_{\hat{v}XY} \circ (v \otimes id_Y) \circ w]_{X \otimes Y}$$
$$\varepsilon([\vartheta|v]_X) = g_X(\vartheta \otimes v)$$
$$\Delta([\vartheta|v]_X) = \sum_j [\vartheta | e_j^{(X)}]_X \otimes [e^j_{(X)} | v]_X$$
$(e_j^{(X)})_j, (e^j_{(X)})_j$ pair of dual bases w.r.t. g_X
$$S([\vartheta|v]_X) = \cdots$$

2) $\mathcal{C} \simeq_\otimes \mathcal{M}^H$ as $\mathbb{K}$-linear additive monoidal categories

<u>Remark:</u> $H \cong \mathrm{Coend}(\mathcal{C},\omega)$ as WHAs always

$$\mathrm{Cat}{\downarrow}\mathrm{Vect}_{\mathbb{K}}^{\otimes *} \underset{\mathcal{M}^-}{\overset{\mathrm{Coend}(-)}{\rightleftarrows}} \mathrm{WHA}_{\mathbb{K}} \quad (\perp)$$

$$\varepsilon: 1_{\mathrm{WHA}_{\mathbb{K}}} \longrightarrow \mathrm{Coend}(\mathcal{M}^-, U)$$
$$\eta: (\mathcal{M}^{\mathrm{Coend}(\mathcal{C},\omega)}, U^{\mathrm{Coend}(\mathcal{C},\omega)}) \longrightarrow 1_{\mathrm{Cat}{\downarrow}\mathrm{Vect}_{\mathbb{K}}^{\otimes *}}$$

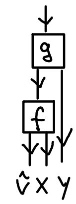
4 Characterization of fusion categories

Q: How exactly does $H = \mathrm{coend}(\mathcal{C}, \omega)$ look like?

observe: lax monoidal structure:

$$\omega_{X,Y} : \underset{\mathrm{Hom}(\hat{V}, \hat{V} \otimes X)}{\omega X} \otimes \underset{\mathrm{Hom}(\hat{V}, \hat{V} \otimes Y)}{\omega Y} \longrightarrow \underset{\mathrm{Hom}(\hat{V}, \hat{V} \otimes (X \otimes Y))}{\omega(X \otimes Y)}$$

$$f \otimes g \longmapsto \alpha_{\hat{V}XY} \circ (f \otimes id_Y) \circ g$$



Use ${}_R\mathcal{M}_R$ structure: $\omega(X \otimes Y) = \omega X \otimes_R \omega Y$, $R = \mathrm{End}(\hat{V})$

If $f : V_\ell \to V_m \otimes X$, $g : V_i \to V_j \otimes X$, then $\omega_{XY}(f \otimes g) = 0$ unless $\ell = j$

idea: $p \in \mathrm{Hom}(V_\ell, V_m \otimes X)$ has source $\sigma(p) = \ell$, target $\tau(p) = m$

and $\omega_{XY}(p \otimes q) = 0$ unless $\sigma(p) = \tau(q)$.

Def: $\mathcal{C}$ multi-fusion category

$M \in |\mathcal{C}|$ generating object, i.e. $\forall j \in I\ \exists m \geq 0 : M^{\otimes m} \cong V_j \oplus$ something

The <u>dimension graph $\mathcal{G}$ of $\mathcal{C}$ w.r.t. M</u> has

vertices: $\mathcal{G}^0 = I$

edges $j \to \ell$: $\mathcal{G}^1_{j\ell} =$ basis of $\mathrm{Hom}(V_\ell, V_j \otimes M)$

notation: $\mathcal{G}^m =$ paths of length $m \geq 1$ in $\mathcal{G}$

Example: $\mathcal{C}_{U_q(sl_2), p}$ with $M = V_1$: $\mathcal{G} =$  (double of A_{p-1})

<u>Prop:</u> 1) There is a WBA

$$H[\mathcal{G}] = \bigoplus_{m \geq 0} \underbrace{(k\mathcal{G}^m)^* \otimes k\mathcal{G}^m}_{H[\mathcal{G}]_0} \ni [p|q]_m,$$

with operations

$$\mu([p|q]_m \otimes [r|s]_n) = \delta_{\sigma(p)\tau(r)}\,\delta_{\sigma(q)\tau(s)}\,[pr|qs]_{m+n}$$ (path algebra of $\mathcal{G} \times \mathcal{G}$ up to identifying $(k\mathcal{G}^m)^* = k\mathcal{G}^m$)

$$\eta(1) = \sum_{i,j \in I} [i|j]_0$$

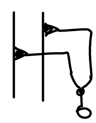
$$\Delta([p|q]_m) = \sum_{r \in \mathcal{G}^m} [p|r]_m \otimes [r|q]_m$$

$$\varepsilon([p|q]_m) = \delta_{p,q}.$$

2) $H[\mathcal{G}]$ is generated as an algebra by $H_0[\mathcal{G}] \cup H[\mathcal{G}]_1$

3) The simple $H[\mathcal{G}]$-comodules are the $k\mathcal{G}^m$, $m \geq 0$: $\beta_{k\mathcal{G}^m}(p) = \sum_{q \in \mathcal{G}^m} q \otimes [q|p]_m$.

4) $k\mathcal{G}^m \hat{\otimes} k\mathcal{G}^\ell \cong k\mathcal{G}^{m+\ell}$ because



$$(p \otimes q) = \begin{cases} p \otimes q & \text{if } \sigma(p) = \tau(q) \\ 0 & \text{else} \end{cases}$$

→ Tensor product of comodules is described by composability of paths

→ The following theorem says that this WBA $H[\mathcal{G}]$ controls the tensor product of all fusion categories:

<u>Theorem:</u> $\mathcal{C}$ multi-fusion category, $H = \mathrm{coend}(\mathcal{C}, \omega)$

1) There is a unique homomorphism of algebras $\pi: H[\mathcal{G}] \to H$ such that

$$\pi([i|j]_0) = [\lambda_i|\lambda_j]_{\mathbb{1}}$$

$$\pi([p|q]_1) = [p|q]_M$$

2) This π is a surjection of WBAs.

→ If we can compute the kernel of π, we have won.

<u>Analogy:</u> $H[\mathcal{G}]$ for a finite graph $\mathcal{G}$ has the following analogy.

<u>Example:</u> $M_q(2) = \mathbb{C}\{a,b,c,d\} \Big/ \begin{pmatrix} ba = qab, & db = qbd, & bc = cb \\ ca = qac, & dc = qcd, & ad - da = (q^{-1} - q)cb \end{pmatrix}$ $\qquad q^2 \neq -1$
(bialgebra)

$SL_q(2) = M_q(2) / (q\det - 1)$, $q\det = da - qbc$ central group-like
(Hopf algebra)

<u>general construction:</u> [Faddeev-Reshetikhin-Takhtajan]

$H[n] := \mathbb{C}\{ t_i^j \mid 1 \leq i,j \leq n \}$ free algebra on n^2 generators

$V = \mathbb{C}^n$

$R : V \otimes V \to V \otimes V$ solution to the QYBE $R_1 R_2 R_1 = R_2 R_1 R_2$ where $\begin{cases} R_1 = R \otimes id \\ R_2 = id \otimes R \end{cases}$

$H[n,R] := H[n] \Big/ \left(\sum_{k\ell} (R_{ij}^{k\ell} t_k^p t_\ell^q - t_i^k t_j^\ell R_{k\ell}^{pq} \;\middle|\; 1 \leq i,j,p,q \leq n \right)$ bialgebra

<u>Example:</u> $n = 2$, $R = \begin{pmatrix} q & & & \\ & 0 & 1 & \\ & 1 & q-q^{-1} & \\ & & & q \end{pmatrix}$ gives $H[2,R] = M_q(2)$

<u>analogy:</u> If $|\mathcal{G}^0| = 1$, $|\mathcal{G}^1| = n$, then $H[\mathcal{G}] = H[n]$ with $[p|q]_1 = t_p^q$ 
n edges

→ We need a weak analogue of the FRT-construction.

→ The purpose of R in the FRT construction is that $R_1, \dots, R_{n-1}$ generate $\operatorname{End}(V^{\otimes n})$ $\forall n \geq 2$

Def: $\mathcal{C}$ multi-fusion category, M generator, $\mathcal{G}$ dimension graph

generating sets $E^{(n)} \subseteq \mathrm{End}(M^{\otimes n})$

$$H[\mathcal{G},E] := H[\mathcal{G}] \Big/ \sum_{p_j \in \mathcal{G}^1} \left([r_1|p_1]_1 \cdots [r_n|p_n]_1 \, f^{(n)}_{p_1 \cdots p_n, q_1 \cdots q_n} - f^{(n)}_{r_1 \cdots r_n, p_1 \cdots p_n} [p_1|q_1]_1 \cdots [p_n|q_n]_1 \right)$$

$r_j, p_j \in \mathcal{G}^1$, $f^{(n)} \in E^{(n)}$, $n \geq 0$ ⟵ coeff. of $\omega(f^{(n)})$

Example: For $\mathcal{C}_{U_q(sl_2),p}$, the relations are

(1) $$\sum_{p_1,p_2 \in \mathcal{G}^1} [r_1|p_1]_1 [r_2|p_2]_1 R_{p_1p_2,q_1q_2} - R_{r_1r_2,p_1p_2} [p_1|q_1]_1 [p_2|q_2]_1$$

with

$$R_{(j\,j-1\,j),(j\,j+1\,j)} = t^{-1/2} \frac{[j]_t [j+2]_t}{[j+1]_t^2} \qquad R_{(j\,j+1\,j),(j\,j+1\,j)} = -\, t^{-1/2} \frac{t^{j+1}}{[j+1]_t}$$

$$R_{(j\,j-1\,j),(j\,j-1\,j)} = t^{-1/2} \frac{t^{-(j+1)}}{[j+1]_t} \qquad R_{(j\,j+1\,j),(j\,j-1\,j)} = t^{-1/2}$$

$$R_{(j\,j\pm1\,j\pm2),(j\,j\pm1\,j\pm2)} = t^{-3/2} \qquad R_{\text{others}} = 0.$$

$(j\,j\pm1\,j\pm2)$: path of length 2 = sequence of 3 vertices

$$[j]_t = \frac{t^j - t^{-j}}{t - t^{-1}}, \qquad t = q^2$$

Thm: a) $H[\mathcal{G}] \longrightarrow H[\mathcal{G},E]$ factors through.

$$\begin{array}{ccc} H[\mathcal{G}] & \longrightarrow & H[\mathcal{G},E] \\ & {\scriptstyle \pi} \searrow & \downarrow {\scriptstyle \bar{\pi}} \\ & & H = \mathrm{Coend}(\mathcal{C},\omega) \end{array}$$

b) A linear map $f \in \mathrm{End}_k((\omega M)^{\hat{\otimes} m}) = \mathrm{End}_k(k\mathcal{G}^m)$ is $H[\mathcal{G},E]$-colinear iff it is H-colinear.

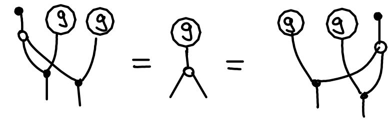
Example: $\mathcal{C}_{U_q(sl_2),p}$, p large

m	$(\omega M)^{\hat{\otimes} m} \in \lvert \mathcal{M}^H \rvert$	$k\mathcal{G}^m \in \lvert \mathcal{M}^{H[\mathcal{G},E]} \rvert$
0	V_0	V_0
1	V_1	V_1
2	$V_2 \oplus V_0$	$V_2 \oplus V_0'$ but $V_0' \not\cong V_0$
3	$V_3 \oplus 2V_1$	$V_3 \oplus 2V_1'$ but $V_1' \not\cong V_1$
4	$V_4 \oplus 3V_2 \oplus 2V_0$	$V_4 \oplus 3V_2' \oplus 2V_0''$ but $V_2' \not\cong V_2$, $V_0'' \not\cong V_0$
$\vdots$		

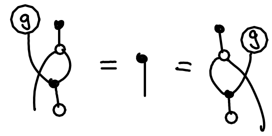
coeff. coalgebras in different degrees

Def: H WBA. Then $g \in H$ is called group-like if

(a)



(b)



Thm: $\ker \bar{\pi} = (1-g \mid g \text{ group-like and } \bar{\pi}(g)=1)$.

Example $\mathcal{C}_{U_q(sl_2),p} \simeq \mathcal{M}^H$, $H = H[\mathcal{G}]/(\text{eq.}(1), \text{eq.}(2))$

(2) $$1 - \sum_{j,\ell \in \mathcal{G}^0} \alpha_j \alpha_\ell \Big(\frac{[\ell+1]_t}{[j+1]_t}[j\,j{+}1\,j \mid \ell\,\ell{+}1\,\ell]_2 - \frac{[\ell+1]_t}{[j]_t}[j\,j{-}1\,j \mid \ell\,\ell{+}1\,\ell]_2$$
$$- \frac{[\ell]_t}{[j+1]_t}[j\,j{+}1\,j \mid \ell\,\ell{-}1\,\ell]_2 + \frac{[\ell]_t}{[j]_t}[j\,j{-}1\,j \mid \ell\,\ell{-}1\,\ell]_2 \Big)$$

where $\alpha_0 = \alpha_{r-2} = 1$

and $\alpha_j = \frac{1}{\sqrt{2}}$ for $1 \leq j \leq t-3$.

Summary: Every multi-fusion category $\mathcal{C}$ is of the form $\mathcal{M}^H$ with the Weak Hopf Algebra $H = H[\mathcal{G}]/(\text{relations A, relations B})$

$\mathcal{G}$: dimension graph of $\mathcal{C}$ w.r.t. a generator M

relations A: generalized 'RTT−TTR' enforce $\mathrm{End}(M^{\otimes m})$, here eq. (1)

relations B: generalized '1−qdet' compare different m., here eq. (2)