Tannaka–Kreĭn Reconstruction and Modular Tensor Categories

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Outline

Modular Categories

Definition ($\approx$ Turaev)

A *modular category* $\mathcal{C}$ is

- ribbon (monoidal, duals, braiding, twist)
- additive ($\to$ all finite biproducts)
- $k = \mathrm{End}(\mathbb{1})$ a field
- finitely semisimple, in particular

$$X \quad \text{simple} \quad \Rightarrow \quad \mathrm{End}(X) \cong k$$

- non-degenerate, *i.e.* the matrix S is invertible,

$S_{j\ell} =$  $\in k \qquad j, \ell$ simple

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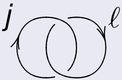

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Modular Categories

Properties

Every modular category $\mathcal{C}$ is

- k-linear, abelian
- $\mathrm{Hom}(X, Y)$ finite-dimensional over k for all $X, Y \in |\mathcal{C}|$
- $tr_X(- \circ -)\colon \mathrm{Hom}(Y, X) \otimes \mathrm{Hom}(X, Y) \to k$ non-degenerate
- $\dim X \neq 0$ for all simple X

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Is every modular category

$$\mathcal{C} \simeq {}_A\mathcal{M}$$

where A is an algebra with extra structure and properties?

Theorem (Etingof–Ostrik)

If $k = \mathbb{C}$ and $\mathcal{C}$ has objects of non-integer Frobenius–Perron dimension, then there is no quasi-bialgebra H such that $\mathcal{C} \simeq {}_H\mathcal{M}$.

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What do we know?

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- $\mathcal{C}$ is abelian, and so
$$\mathcal{C} \hookrightarrow {}_A\mathcal{M}$$
is a full subcategory of ${}_A\mathcal{M}$ for some ring A [Freyd–Mitchell] (if $\mathcal{C}$ is *e.g.* essentially small).
- $\mathcal{C}$ is finitely semisimple, and so
$$A = M_{n_1}(k) \oplus M_{n_2}(k) \oplus \cdots$$
- Why don't we take this A and compute the extra structure?
- $n_1, n_2, \ldots = ?$ Need a fiber functor $\mathcal{C} \to {}_k\mathcal{M}$.

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Not in general.
Either strong monoidal or into ${}_k\mathcal{M}$, but not both.

Which forgetful functor?

Result (Hayashi, Hai)

Define

$$\hat{V} = \bigoplus_{j \in I} j \quad \text{(universal object)}, \qquad R = \mathrm{End}(\hat{V}).$$

The *short* canonical functor

$$\hat{\omega} \colon \mathcal{C} \to {}_R\mathcal{M}_R, \quad X \mapsto \mathrm{Hom}(\hat{V}, \hat{V} \otimes X),$$

is strong monoidal.

- Get a bialgebroid.
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R is commutative and separable, and so the *long* canonical functor

$$\omega\colon \mathcal{C} \to {}_k\mathcal{M}, \quad X \mapsto \mathrm{Hom}(\hat{V}, \hat{V} \otimes X),$$

has a *separable Frobenius structure* and is the forgetful functor of a Weak Hopf Algebra.

Definition

A functor $(\omega, \omega_{X,Y}, \omega_0, \omega^{X,Y}, \omega^0)\colon \mathcal{C} \to \mathcal{C}'$ with *separable Frobenius structure* is

- lax monoidal $(\omega, \omega_{X,Y}, \omega_0)$,
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Reconstruction

Coalgebra structure

Reconstruct linear maps from the universal property of the coend $H = \textbf{coend}(\mathcal{C}, \omega)$:

$$\begin{tikzcd}
\omega X \arrow[r, "\delta^{\omega}_{X}"] \arrow[d, "\delta^{\omega}_{X}"'] & \omega X \otimes H \arrow[dd, "\omega X \otimes \Delta"] \\
\omega X \otimes H \arrow[d, "\delta^{\omega}_{X} \otimes H"'] & \\
(\omega X \otimes H) \otimes H \arrow[r, "\alpha_{\omega X, H, H}"'] & \omega X \otimes (H \otimes H)
\end{tikzcd}$$

Reconstruction

Coalgebra structure (cont'd)

$$\begin{tikzcd}
\omega X \arrow[r, "\delta^{\omega}_{X}"] \arrow[dr, "\rho^{-1}_{\omega X}"'] & \omega X \otimes H \arrow[d, "\omega X \otimes \varepsilon"] \\
 & \omega X \otimes k
\end{tikzcd}$$

(H, Δ, ε) is a finite-dimensional split cosemisimple counital coassociative coalgebra over k,

$$H \cong \bigoplus_{j \in I} (\omega j)^* \otimes \omega j.$$

Reconstruction

Algebra structure

Reconstruct

$$\begin{tikzcd}
\omega X \otimes \omega Y \arrow[r, "\delta^{\omega \otimes \omega}_{X,Y}"] \arrow[d, "\omega_{X,Y}"'] & (\omega X \otimes \omega Y) \otimes (H \otimes H) \arrow[dd, "\omega X \otimes \omega Y \otimes \mu"] \\
\omega(X \otimes Y) \arrow[d, "\delta^{\omega}_{X \otimes Y}"'] & \\
\omega(X \otimes Y) \otimes H \arrow[r, "\omega^{X,Y} \otimes H"'] & (\omega X \otimes \omega Y) \otimes H
\end{tikzcd}$$

Reconstruction

Algebra structure (cont'd)

$$\begin{tikzcd}
k \arrow[r, "\delta^{\omega^{\otimes 0}}"] \arrow[d, "\omega_0"'] & k \otimes k \arrow[dd, "k\otimes\eta"] \\
\omega(\mathbb{1}) \arrow[d, "\delta^{\omega}_{\mathbb{1}}"'] & \\
\omega(\mathbb{1}) \otimes H \arrow[r, "\omega^0 \otimes H"'] & k \otimes H.
\end{tikzcd}$$

(H, μ, η) is a unital associative algebra.

Result

Definition

A Weak Hopf Algebra *H* over *k* is called *comodular* if *H* is

- finite-dimensional
- split cosemisimple
- coribbon (coquasitriangular, universal ribbon form)
- $H_s \cap H_t \cong k$
- *weakly cofactorizable*, *i.e.* every linear form $\varphi\colon T_H \to k$ is of the form $\varphi(-) = q(- \otimes x)$ for some $x \in T_H$. Here

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  - $T_H = \operatorname{span}$

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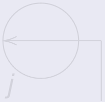
- $T_H = \mathrm{span}_k\Big\{$ , $\quad j \in I \Big\}$.

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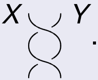
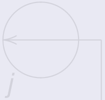
    - $q\colon H \otimes H \to k$ is reconstructed from $Q_{X,Y} =$ .
    - $T_H = \mathrm{span}_k \{$  $, \quad j \in I \}$.

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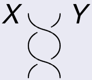
  - $q\colon H \otimes H \to k$ is reconstructed from $Q_{X,Y} =$ 

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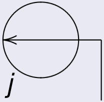
  - $T_H = \mathrm{span}_k \Big\{$ 

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Theorem

- *Every modular category $\mathcal{C}$ is of the form*

  $$\mathcal{C} \simeq \mathcal{M}^H$$

  *where $H =$ **coend**$(\mathcal{C}, \omega)$ is a comodular Weak Hopf Algebra.*
- *If H is a comodular Weak Hopf Algebra, then $\mathcal{M}^H$ is modular.*

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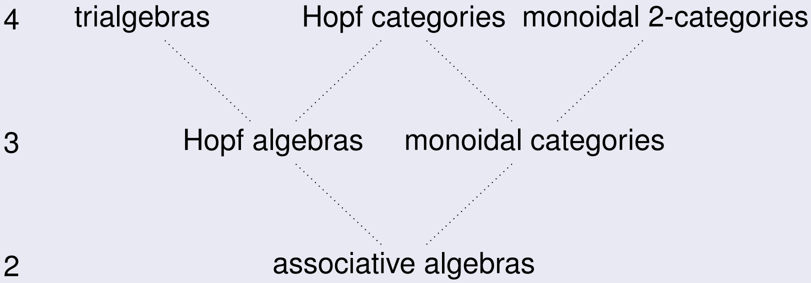
Application in Topology

Categorification [Crane–Frenkel]

Algebraic structures used for combinatorial invariants of low-dimensional (smooth) manifolds:



4 trialgebras Hopf categories monoidal 2-categories

3 Hopf algebras monoidal categories

2 associative algebras

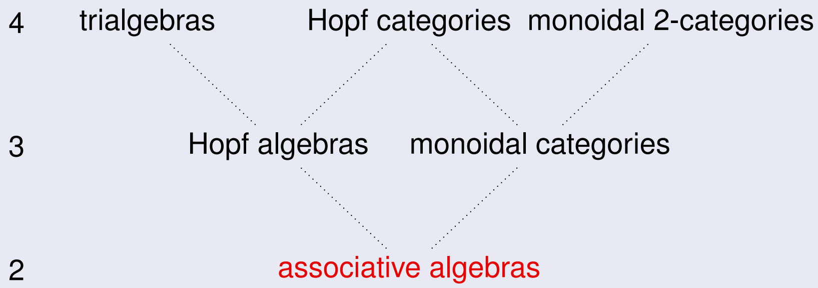
Application in Topology

d=2

Strongly separable associative algebra [Fukuma–Hosono–Kawai]

Categ

Algeb of low-dimensional (smooth) manifolds.



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Application in Topology

d=3

Finitely semisimple spherical category
[Turaev–Viro, Barrett–Westbury]

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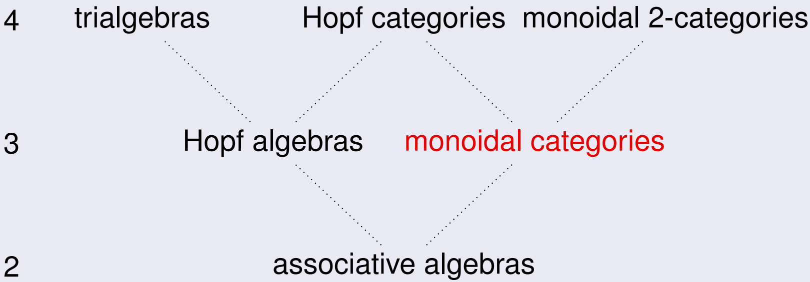

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Finite-dimensional involutory Hopf algebra [Kuperberg]

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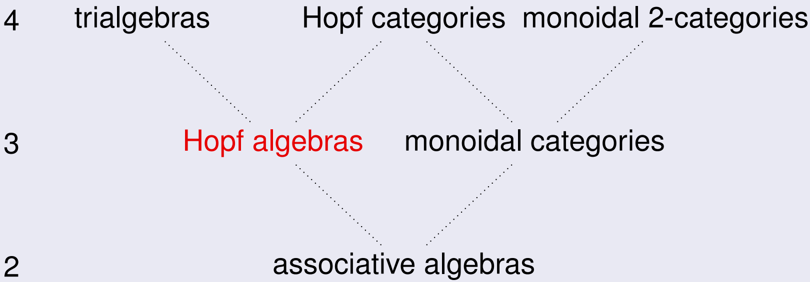

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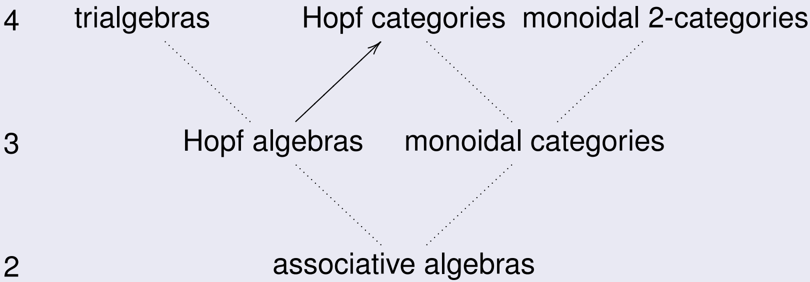

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$H \mapsto {}_H\mathcal{M}$ [Barrett–Westbury]

Categ

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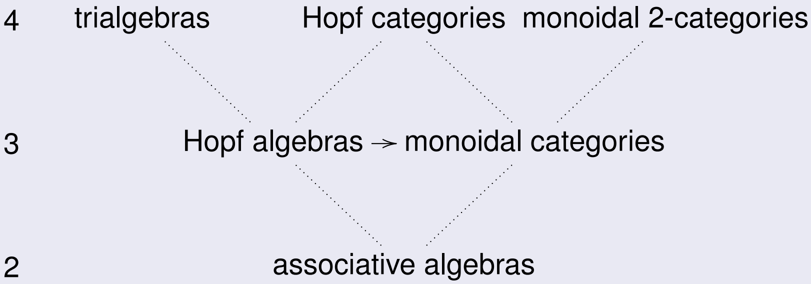


4 trialgebras Hopf categories monoidal 2-categories

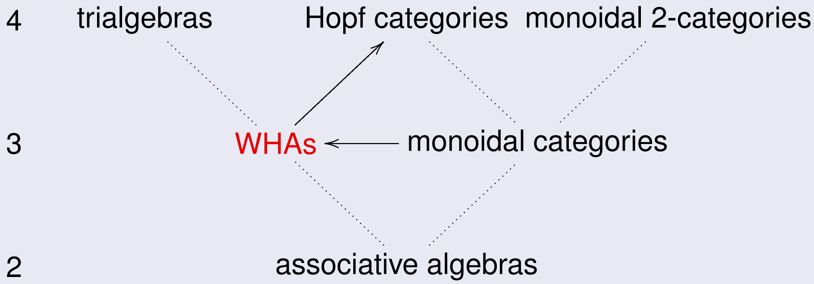
3 Hopf algebras → monoidal categories

2 associative algebras

Application in Topology

Categorification [Crane–Frenkel]

Algebraic structures used for combinatorial invariants of low-dimensional (smooth) manifolds:



4 trialgebras Hopf categories monoidal 2-categories

3 WHAs ← monoidal categories

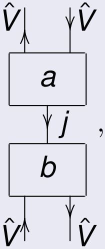
2 associative algebras

The reconstructed Weak Hopf Algebra

The reconstructed Weak Hopf Algebra

$$H = \bigoplus_{j \in I} \mathrm{Hom}(j, \hat{V}^* \otimes \hat{V}) \otimes \mathrm{Hom}(\hat{V}^* \otimes \hat{V}, j).$$

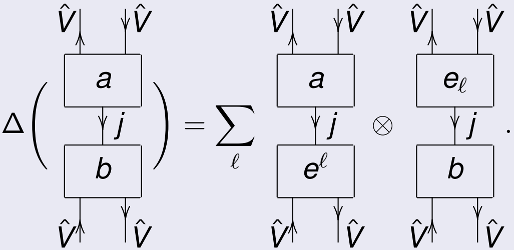
Basis:



$$\left\{ \quad , \qquad j \in I,\ a \in \mathrm{Hom}(\hat{V}^* \otimes \hat{V}, j),\ b \in \mathrm{Hom}(j, \hat{V}^* \otimes \hat{V}) \right\}.$$

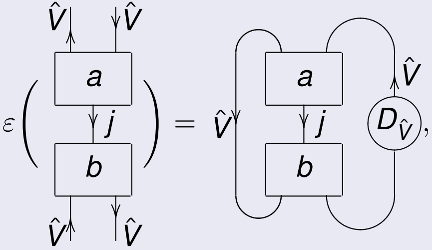
Coalgebra structure

Coalgebra structure

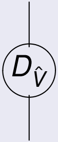


Coalgebra structure

Coalgebra structure (cont'd)



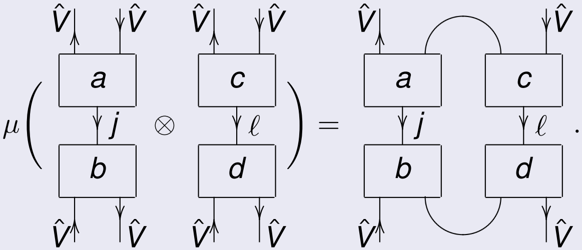
where



$$= \sum_{j\in I} \mathrm{id}_j \, (\dim j)^{-1} \colon \hat{V} \to \hat{V}.$$

Algebra structure

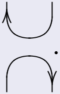
Algebra structure



Algebra structure

Algebra structure (cont'd)

$\eta(1) =$



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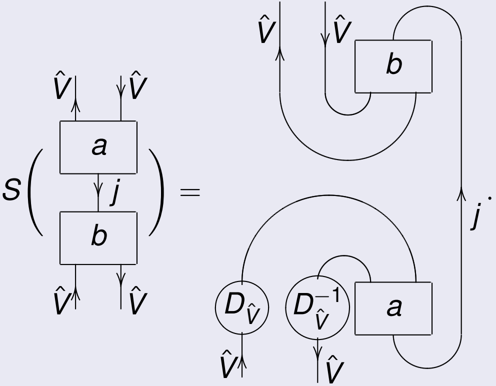
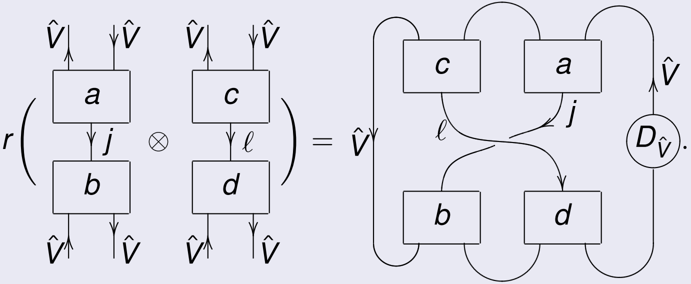
Antipode

Antipode



Coquasitriangular structure

Coquasitriangular structure



Universal ribbon form

Universal ribbon form