2-dimensional extended Topological Quantum Field Theories and Khovanov homology

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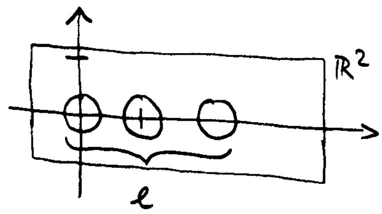
1 Cobordisms and Topological Quantum Field Theories (TQFTs)
2 Manifolds with corners and extended TQFTs
3 Combinatorial construction
4 Khovanov homology from links to tangles

1 Cobordisms and TQFTs

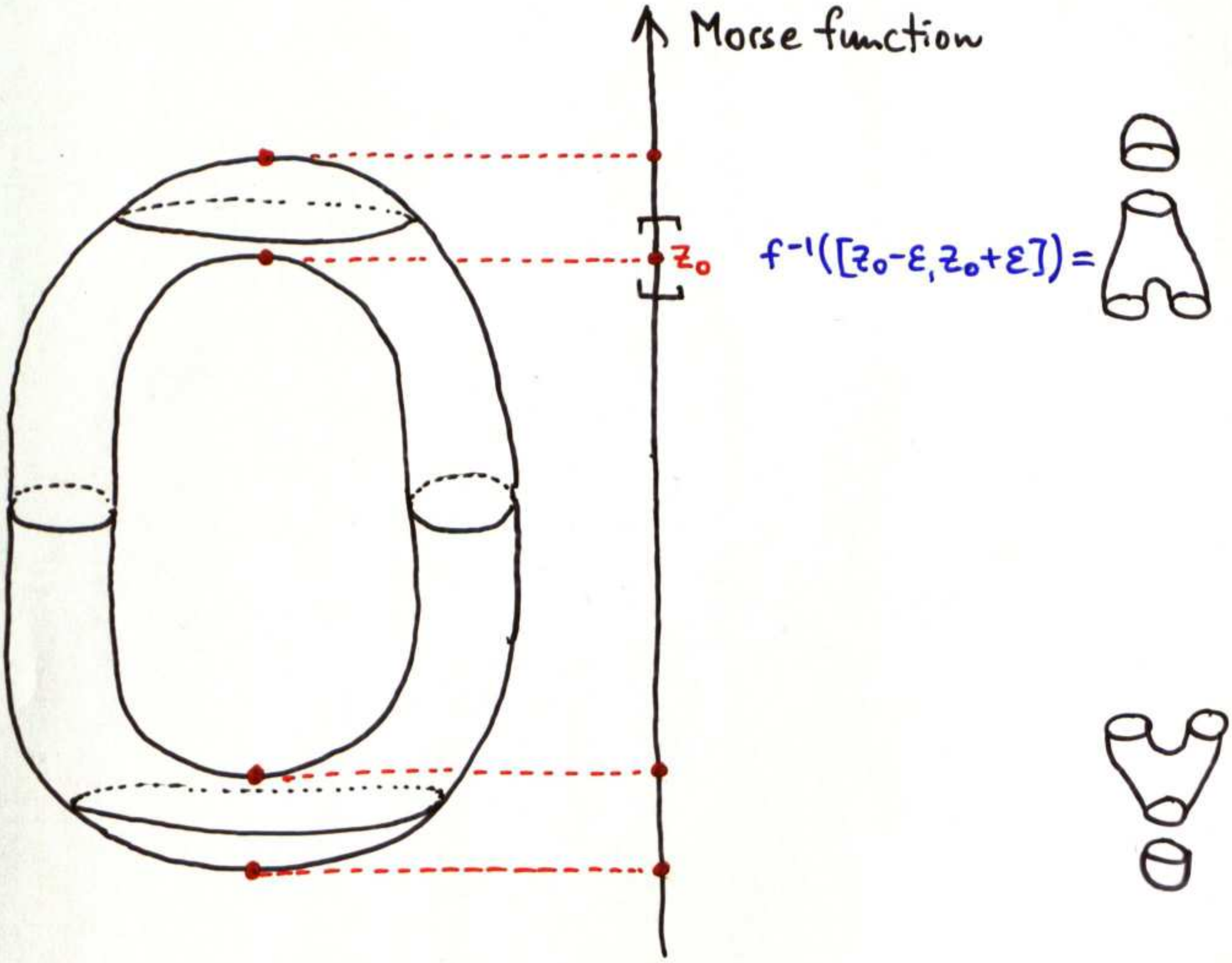
- try to classify cobordisms
- use vector spaces and linear maps to compute invariants
- translate from topology to algebra

$\ell \in \mathbb{N}_0$, $\quad \Sigma_\ell := \dot{\bigcup}_{j=0}^{\ell-1} \partial B_{1/4}((j,0)) \subseteq \mathbb{R}^2$ (induced orientation)



Σ_ℓ^* (opposite orientation)

$T^2 \subseteq \mathbb{R}^3$



A 2-dimensional TQFT sends

(1) closed 1-manifolds to k-vector spaces

 $\longmapsto$ C

 $\longmapsto$ $C \otimes C$

$\emptyset$ $\longmapsto$ k

(2) 2-dimensional cobordisms to k-linear maps

 $\longmapsto$ $k \xrightarrow{\eta} C$

 $\longmapsto$ $C \otimes C \xrightarrow{\mu} C$

def: The category 2Cob of 2-dimensional cobordisms consists of

objects: $\ell \in \mathbb{N}_0$

morphisms: $[(M,\psi_1,\psi_2)] : \ell \longrightarrow \ell'$

M compact oriented smooth 2-manifold

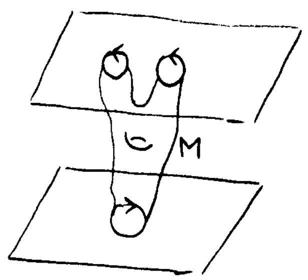
$$\left.\begin{array}{l}\psi_1 : \Sigma_\ell^* \longrightarrow \psi_1(\Sigma_\ell^*) \subseteq \partial M \\ \psi_2 : \Sigma_{\ell'} \longrightarrow \psi_2(\Sigma_{\ell'}) \subseteq \partial M\end{array}\right\} \text{ orientation preserving diffeos}$$

such that $\partial M = \psi_1(\Sigma_\ell^*) \,\dot\cup\, \psi_2(\Sigma_{\ell'})$

$(M,\psi_1,\psi_2) \sim (N,\varphi_1,\varphi_2)$ iff $\exists f : M \longrightarrow N$ orientation preserving diffeomorphism such that

$$\begin{tikzcd} & M \arrow[d, "f"] & \\ \Sigma_\ell^* \arrow[ur, "\psi_1"] \arrow[r, "\varphi_1"'] & N & \Sigma_{\ell'} \arrow[ul, "\psi_2"'] \arrow[l, "\varphi_2"] \end{tikzcd}$$

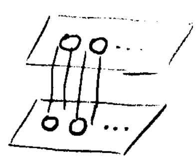
picture: Σ_2^*



Σ_1

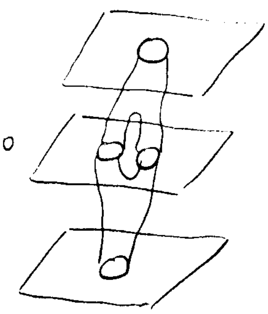
up to diffeomorphism rel ∂ (<u>not</u> ambient isotopy!)

identity: id_{Σ_ℓ}

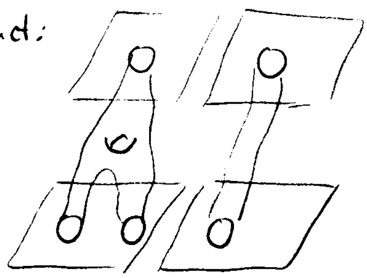


Composition:

$\downarrow \circ$

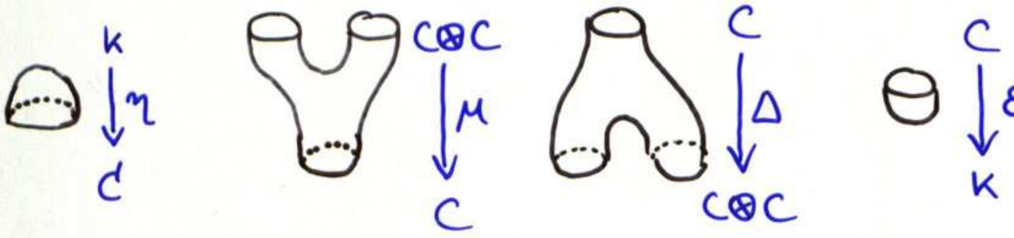


tensor product:

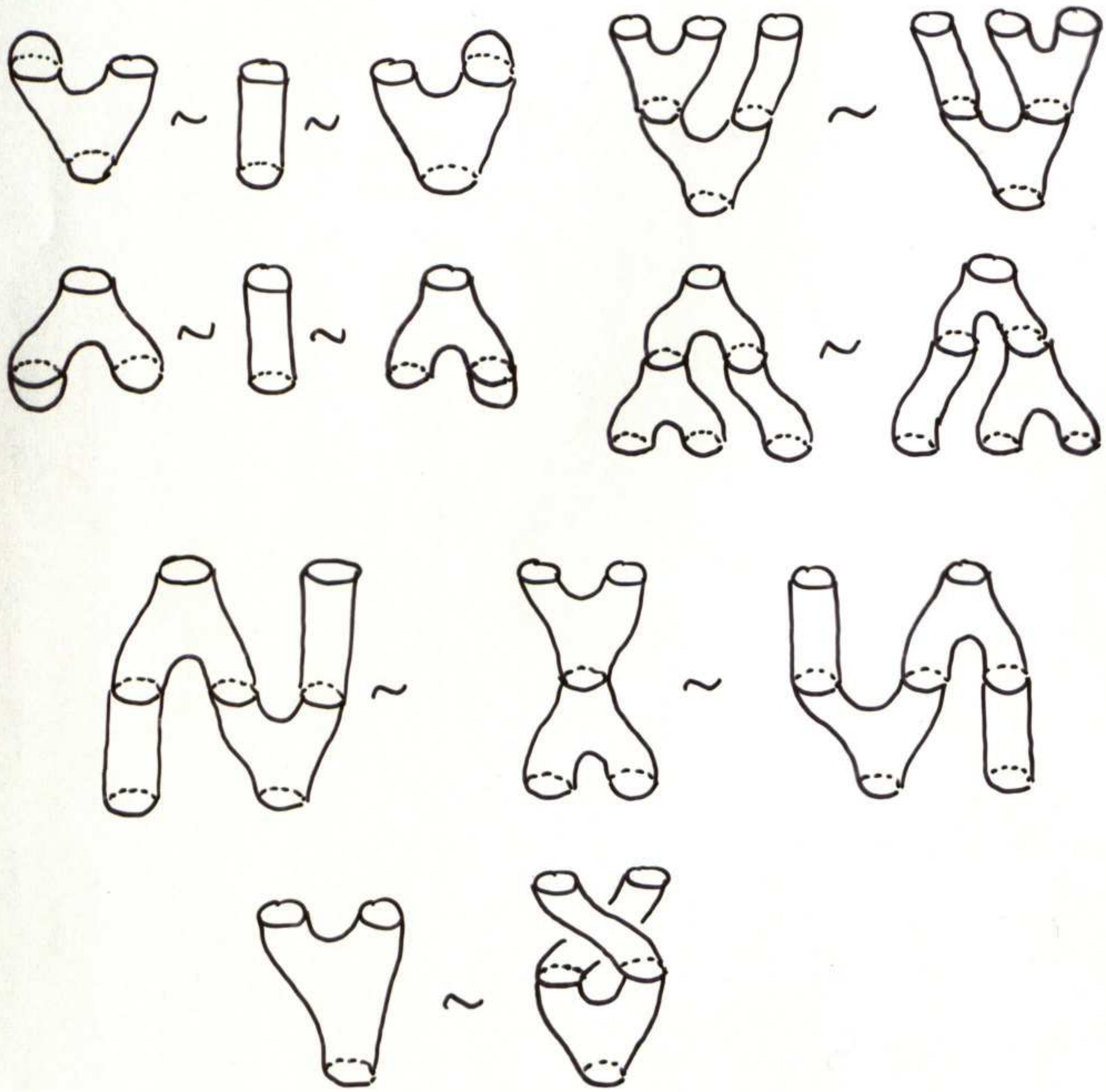


def: A 2-dimensional TQFT is a symmetric monoidal functor

$Z : 2\mathrm{Cob} \longrightarrow \mathrm{Vect}_K$



The following morphisms of 2Cob are equivalent:



def: A Frobenius algebra $(C, \mu, \eta, \Delta, \varepsilon)$ is a k-vector space C with linear maps

$$\eta: k \to C, \quad \mu: C\otimes C \to C, \quad \Delta: C \to C\otimes C, \quad \varepsilon: C \to k$$

such that

(a) (C, μ, η) is a unital associative algebra

(b) (C, Δ, ε) is a counital coassociative coalgebra, i.e.

(i) $(\mathrm{id}\otimes\varepsilon)\circ\Delta = \mathrm{id}_C = (\varepsilon\otimes\mathrm{id})\circ\Delta$

(ii) $(\Delta\otimes\mathrm{id})\circ\Delta = (\mathrm{id}\otimes\Delta)\circ\Delta$

(c) $(\mathrm{id}\otimes\mu)\circ(\Delta\otimes\mathrm{id}) = \Delta\circ\mu = (\mu\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta)$

It is called commutative if

$$\mu\circ\tau = \mu$$

It is called symmetric if $\qquad \tau(a\otimes b) = b\otimes a$

$$\varepsilon\circ\mu\circ\tau = \varepsilon\circ\mu$$

thm: [Abrams, Sawin]

The category of 2d TQFTs is equivalent to the category of commutative Frobenius algebras:

$$\mathrm{SymMonFunct}(2\mathrm{Cob}, \mathrm{Vect}_k) \simeq \mathrm{ComFrob}$$

example: $C = k[x]/(x^2)$ [Khovanov]

$\Delta(1) = 1\otimes x + x\otimes 1$ $\qquad \varepsilon(1) = 0$

$\Delta(x) = x\otimes x$ $\qquad \varepsilon(x) = 1$

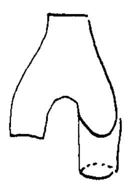
$h, t \in k$

$C_{h,t} = k[x]/(x^2 - hx - t)$

$\Delta(1) = 1\otimes x + x\otimes 1 - h 1\otimes 1 \qquad \varepsilon(1) = 0$

$\Delta(x) = x\otimes x + t 1\otimes 1 \qquad \varepsilon(x) = 0$

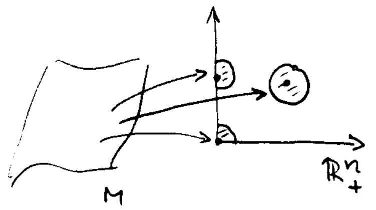
aim: Extend the notion of cobordisms to manifolds whose boundaries are —'s or O's.



A 2-dimensional ⟨2⟩-manifold:

M $\quad$ ∂M $\quad$ $\partial_0 M$ "black" $\quad$ $\partial_1 M$ "white" $\quad$ $\partial_0 M \cap \partial_1 M$ "corners" remark: A smooth manifold with corners M has coordinate systems $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ such that the charts are homeomorphisms

$$\varphi_\alpha : U_\alpha \longrightarrow \varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n_+ := [0,\infty)^n$$



and transition functions

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

are restrictions to $\mathbb{R}^n_+$ of diffeomorphisms.

def: $p \in U_\alpha \subseteq M$

$$c(p) = \#\{\, i \mid (\varphi_\alpha(p))_i = 0 \,\} \qquad \text{(number of zero coefficients)}$$

A connected face is the closure of a component of $\{p \in M \mid c(p) = 1\}$

A face is a union of pairwise disjoint connected faces.

A manifold with faces is a smooth manifold with corners such that each $p \in M$ is contained in $c(p)$ different connected faces

A <2>-manifold $(M, \partial_0 M, \partial_1 M)$ is a manifold with faces M with faces $\partial_0 M, \partial_1 M$ such that

(i) $\partial M = \partial_0 M \cup \partial_1 M$

(ii) $\partial_0 M \cap \partial_1 M$ is a face of both $\partial_0 M$ and $\partial_1 M$.

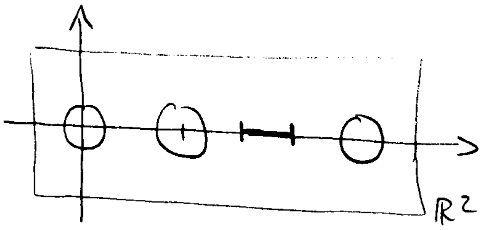
$\ell \in \mathbb{N}_0$

$\underline{n} = (n_0, \dots, n_{\ell-1}) \in \{0,1\}^\ell$

$$\Sigma_{\underline{n}} := \dot{\bigcup_{j=0}^{\ell-1}} \Sigma^{(n_j)}((j,0)) \subseteq \mathbb{R}^2 \qquad \Sigma^{(0)}(x,y) := \partial B_{1/4}(x,y)$$

$$\Sigma^{(1)}(x,y) := [x - 1/4, x + 1/4] \times \{y\}$$

$\Sigma_{0,0,1,0} =$



def: The category $2\mathrm{Cob}^{\mathrm{ext}}$ of 2-dimensional open-closed cobordisms consists of

objects: $\underline{n} \in \{0,1\}^\ell$, $\ell \in \mathbb{N}_0$

morphisms: $[(M, \psi_1, \psi_2)] : \underline{n} \longrightarrow \underline{n}'$

M compact oriented 2-dimensional $\langle 2 \rangle$-manifold

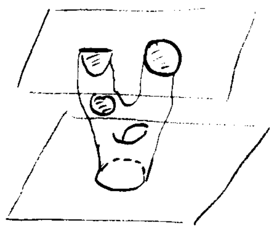
$$\left.\begin{array}{l} \psi_1 : \Sigma^*_{\underline{n}} \longrightarrow \psi_1(\Sigma^*_{\underline{n}}) \subseteq \partial_0 M \\ \psi_2 : \Sigma_{\underline{n}'} \longrightarrow \psi_2(\Sigma_{\underline{n}'}) \subseteq \partial_0 M \end{array}\right\} \text{orientation-preserving diffeo}$$

such that

$$\partial_0 M = \psi_1(\Sigma^*_{\underline{n}}) \,\dot{\cup}\, \psi_2(\Sigma_{\underline{n}'})$$

"$\sim$" as above.

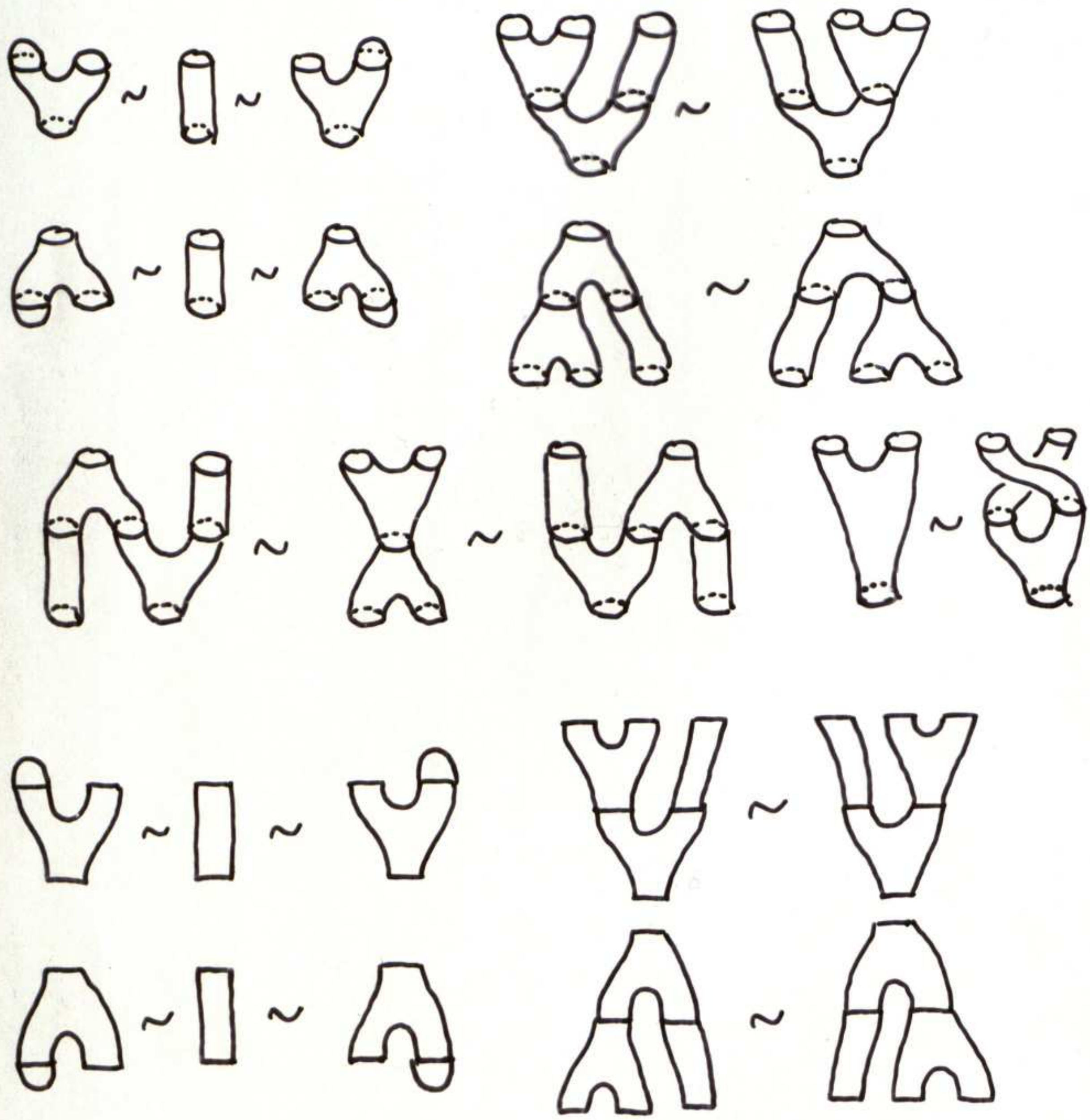
picture:

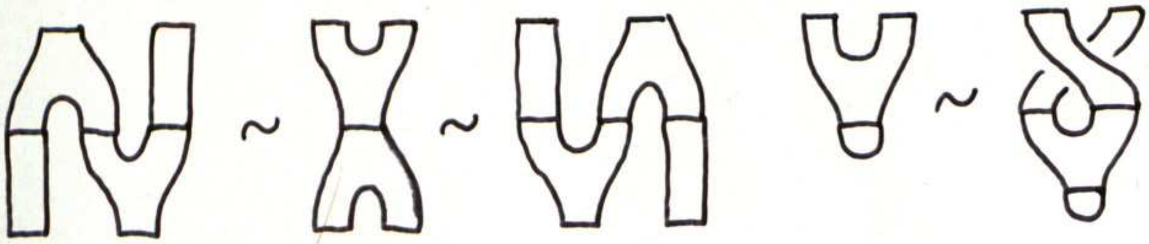


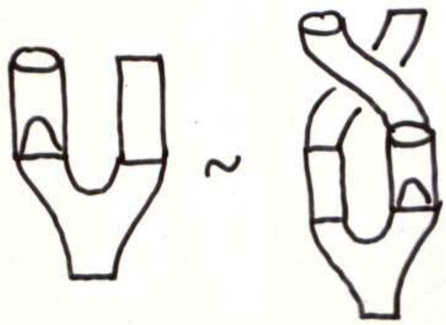
def: A 2-dimensional open-closed TQFT is a symmetric monoidal functor $Z : 2\mathrm{Cob}^{\mathrm{ext}} \longrightarrow \mathrm{Vect}_K$.

η_c μ_c Δ_c ε_c η_A μ_A Δ_A ε_A ι^* ι

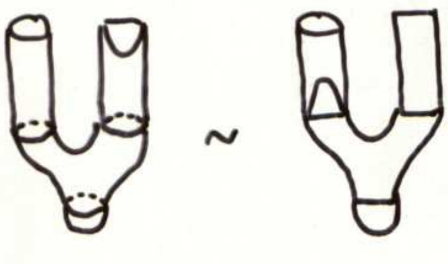
The following morphisms of $2\mathrm{Cob}^{ext}$ are equivalent:



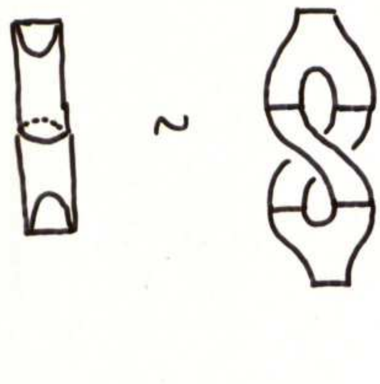




Knowledge



duality



Cardy condition

<u>def:</u> A <u>Knowledgeable Frobenius algebra</u> (A, C, ι, ι^*) consists of

(a) a symmetric Frobenius algebra $(A, \mu_A, \eta_A, \Delta_A, \varepsilon_A)$
(b) a commutative Frobenius algebra $(C, \mu_C, \eta_C, \Delta_C, \varepsilon_C)$
(c) linear maps $\iota: C \to A$, $\iota^*: A \to C$

such that

(i) ι is a homomorphism of algebras
(ii) $\mu_A \circ (\iota \otimes id_A) = \mu_A \circ \tau \circ (\iota \otimes id_A)$
(iii) $\varepsilon_C \circ \mu_C \circ (id_C \otimes \iota^*) = \varepsilon_A \circ \mu_A \circ (\iota \otimes id_A)$
(iv) $\iota \circ \iota^* = \mu_A \circ \tau \circ \Delta_A$

Thm: [AL, HP]
The category $2Cob^{ext}$ is the free strict symmetric monoidal category generated by a Knowledgeable Frobenius algebra object.

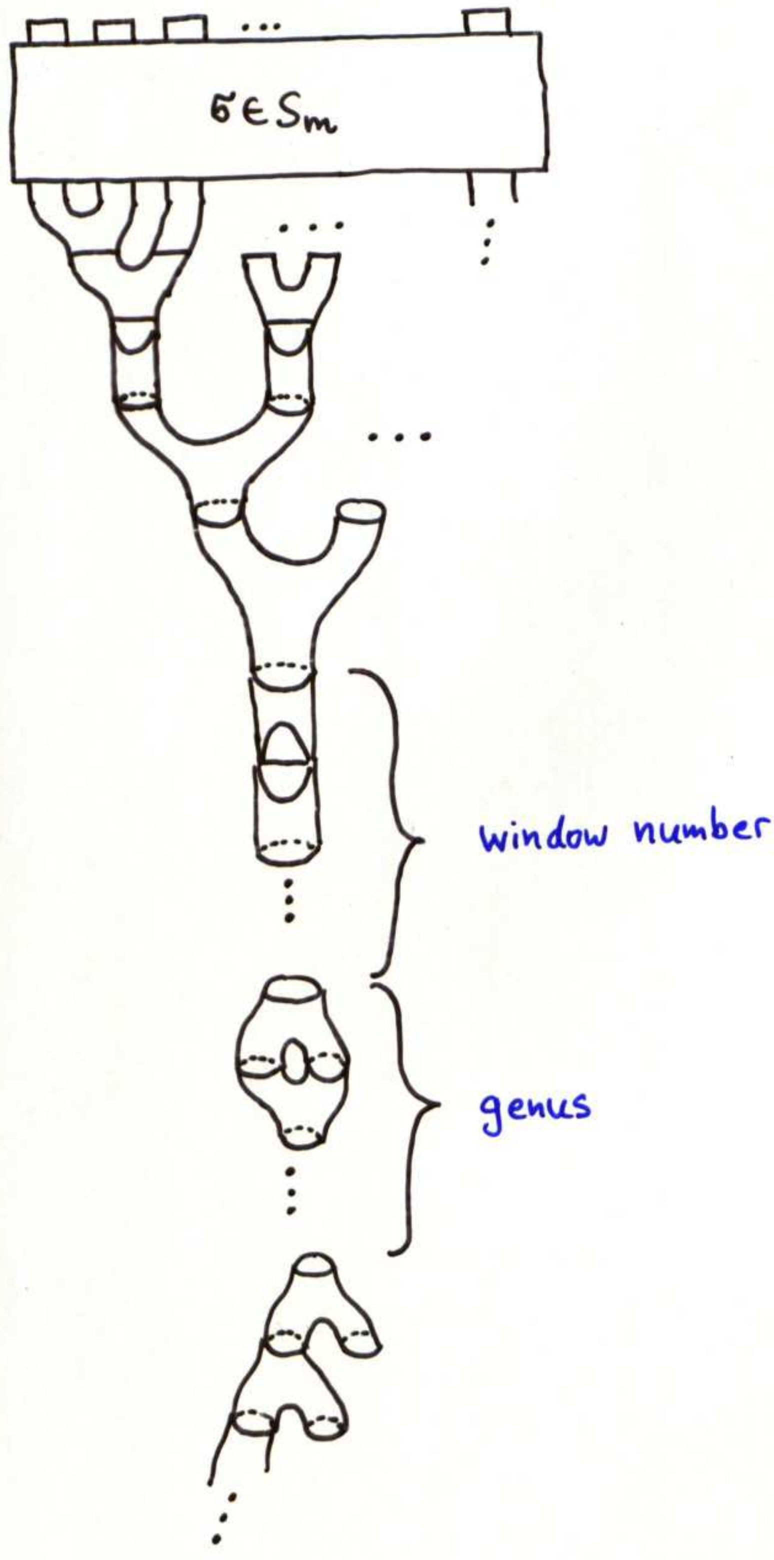
<u>Cor:</u>

$$\mathrm{SymMonFunct}\,[2Cob^{ext}, Vect_K] \simeq K\text{-}Frob$$

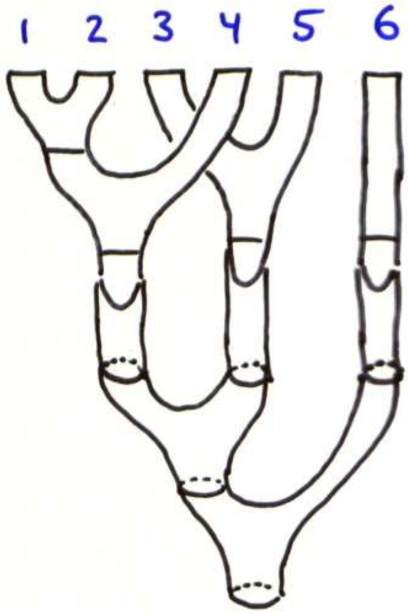
<u>Proof:</u> (1) Find generators for the morphisms of $2Cob^{ext}$ (Morse thy)
(2) relations are necessary, i.e. induced by '$\sim$'
(3) relations are sufficient (normal form)
(4) find invariants:
genus,
window number,
boundary permutation.

Normal form of an open-closed cobordism

$$M: \underbrace{(1, \dots, 1)}_{m} \to \underbrace{(0, \dots, 0)}_{\ell}$$



$m=6$



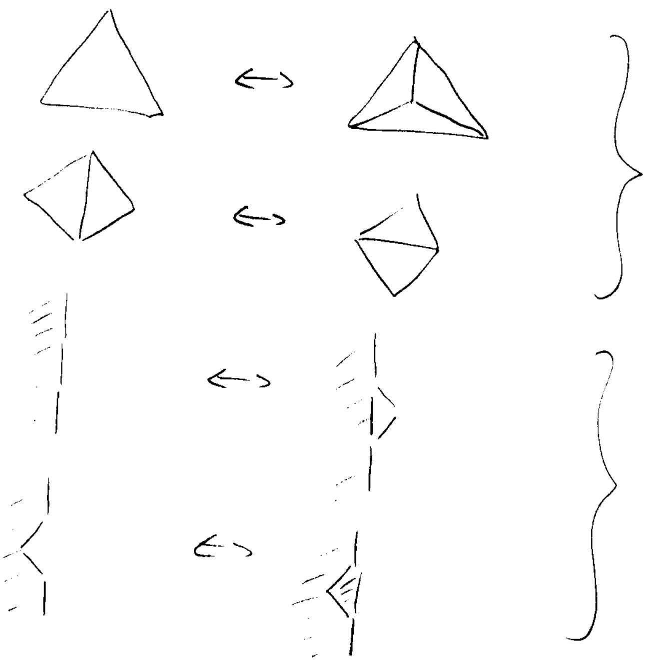
$\sigma = (124)(35) \in S_6$

boundary permutation

thm: Any two triangulations of a compact smooth 2-manifold are related by a finite sequence of the following moves:

1-3:

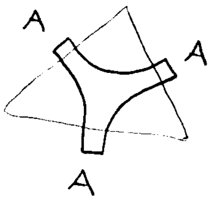
2-2:



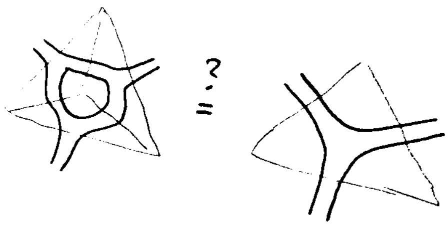
interior

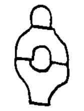
boundary

idea:



question:



def: $a := \mu_A \circ \Delta_A \circ \eta_A(1) = Z\big($  $\big)$ window element

def: A k-algebra A is called <u>strongly separable</u> if the canonical bilinear form

$$a \otimes b \longmapsto tr_A(L_a \circ L_b)$$

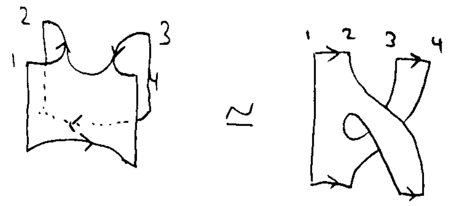
where $L_a : A \to A$, $c \mapsto ac$, is non degenerate.

prop: A symmetric Frobenius algebra $(A, \mu_A, \eta_A, \Delta_A, \varepsilon_A)$ is strongly separable iff the window element is a unit, i.e. a^{-1} exists.

note: $a^{-1} \cdot Z\big($  $\big) = Z\big($  $\big)$ "removes holes"

From links to tangles

- use of TQFTs with (non-extended) cobordisms limits the construction to links rather than tangles
- $\times$ would give an extended cobordism



$$A \otimes A \xrightarrow{(\mu_A \otimes id_A)\circ(id_A \otimes \tau)\circ(\Delta_A \otimes id_A)} A \otimes A$$

- for a (p,q)-tangle T obtain a complex $[\![T]\!]$ of $(A^{\otimes p}, A^{\otimes q})$-bimodules
- $[\![\downarrow]\!] = 0 \to A \to 0$
- $[\![O]\!] = 0 \to C \to 0$
- $[\![\times]\!] = 0 \to A \otimes A \to A \otimes A \to 0$

thm: [AL, HP]

a) For every K-Frob (A, C, ι, ι^*) such that C satisfies $S, T, 4Tu$ relations, there is a tangle homology theory

b) If A, C are strongly separable, tangle composition is $\otimes_{A^{\otimes p}}$ of complexes

c) If C is not strongly separable and A abelian, tangle composition factors through $\otimes_{A^{\otimes p}}$, but another map $A = C \xrightarrow{\iota^*} C$ needs to be applied wherever a circle is formed.

→ We understand which aspects of Khovanov homology can be computed locally and which cannot

<u>Why is this interesting:</u>

<u>thm:</u> [Jacobsson]
Khovanov's chain complexes, viewed in the homotopy category, given an invariant of 2-knots (up to a sign).

<u>thm:</u> [Baez, Langford]
The category of 2-tangles is the free semistrict braided monoidal 2-category with weak duals generated by an unframed selfdual object.

<u>thm:</u> [AL, HP]
We have an example (over $\mathbb{F}_2$).

generator $\bullet$

its identity $\downarrow$

$\rightarrow$ we needed a tangle extension

$C_{h,t} = k[x]/_{(x^2 - hx - t)}$

$\Delta(1) = 1\otimes x + x\otimes 1 - h1\otimes 1$ $\quad\varepsilon(1) = 0$

$\Delta(x) = x\otimes x + t1\otimes 1$ $\quad\varepsilon(x) = 1$

$A = M_{n\times n}(k)$ $\quad \{e_{pq}\}_{1\le p,q\le n}$

$\Delta(e_{pq}) = \alpha\sum_{r=1}^{n} e_{pr}\otimes e_{rq}$ $\quad \alpha\in k\setminus\{0\}$

$\varepsilon(e_{pq}) = \alpha^{-1}\delta_{pq}$

● $C = k$, $= Z(A)$ $\quad \Delta(1) = \alpha^2 1\otimes 1$ $\quad \iota(1) = \sum_{r=1}^{n} e_{rr}$

$\varepsilon(1) = \alpha^{-1}$ $\quad \iota^*(e_{pq}) = \alpha\,\delta_{pq}$

char $k = p \geq 2$

$A = k[y]/_{(y^p)}$ $\quad \Delta(y^\ell) = \sum_{j=0}^{p-1-\ell} y^{j+\ell}\otimes y^{p-1-j}$ $\quad \forall \ell\in\{0\ldots p-1\}$

$\varepsilon(y^{p-1}) = 1$

$\varepsilon(y^\ell) = 0 \quad \forall\ \ell\in\{0\ldots p-2\}$

$C_{0,0}$

● $\iota(1) = 1$ $\quad \iota^*(y^{p-1}) = 1$

$\iota(x) = 0$ $\quad \iota^*(y^\ell) = 0 \quad \forall\ \ell\in\{0\ldots p-2\}$
