Proof of Transience of Simple Symmetric RW in $\mathbb{Z}^3$

$X_n = \sum_{j=1}^{n} Y_j$. Each of the steps Y_j heads E, W, N, S, Up or Down.

$X_{2n} = \vec{0}$ iff #steps E = #steps W = i, #steps N = #steps S = j
and #steps U = #steps D = k.

Calculating this latter probability using the multinomial distribution gives:

$$P_{\vec 0}(X_{2n} = \vec{0}) = \sum_{\substack{i+j+k=n \\ i,j,k \geq 0}} \binom{2n}{i\ j\ k\ i\ j\ k} \left(\frac{1}{6}\right)^{2n}$$

$$= \sum_{\substack{i+j+k=n \\ i,j,k \geq 0}} \frac{2n!}{(i!\,j!\,k!)^2} \left(\frac{1}{6}\right)^{2n}$$

$$= \binom{2n}{n}\left(\frac{1}{2}\right)^{2n} \sum_{\substack{i+j+k=n \\ i,j,k \geq 0}} \underbrace{\binom{n}{i\ j\ k}^2 \left(\frac{1}{3}\right)^{2n}}_{p^{(n)}(i,j,k)^2}$$

$\sum_{\substack{i+j+k=n \\ i,j,k \geq 0}} p^{(n)}(i,j,k) = 1$ as they are the multinomial probabilities with n trials and 3 equally likely outcomes.

$$\therefore\ P_{\vec 0}(X_{2n} = \vec{0}) \leq \binom{2n}{n}\left(\frac{1}{2}\right)^{2n} \max_{\substack{i,j,k \geq 0 \\ i+j+k=n}} \binom{n}{i\ j\ k}\frac{1}{3^n} \times \underbrace{\sum_{\substack{i,j,k \geq 0 \\ i+j+k=n}} p^{(n)}(i,j,k)}_{=1}$$

Consider minimizing $i!\,j!\,k!$ for $i+j+k=n$.
If say $i < \lfloor n/3 \rfloor$ then j (say) must be $> \lfloor n/3 \rfloor$ and increasing i by 1 and decreasing j by 1 will decrease $i!\,j!\,k!$ by a multiplicative factor of $\frac{i+1}{j} < 1$.

Hence the minimum is attained when $i,j,k \geq \lfloor \frac{n}{3} \rfloor$ and similarly $i,j,k \leq \lceil n/3 \rceil$.

So if $\{S_n\}$ is a ssrw on $\mathbb{Z}$, for some n, $\lfloor \frac{n}{3} \rfloor \le i_n, j_n, k_n \le \lceil \frac{n}{3} \rceil$, $i_n + j_n + k_n = n$ we have

$$P_{\vec{0}}(X_{2n} = \vec{0}) \le P(S_{2n} = 0) \frac{n!}{i_n!\, j_n!\, k_n!} \frac{1}{3^n}.$$

$$\le \frac{c}{\sqrt{n}} \frac{n^n \sqrt{n}\, e^{-n}}{i_n^{i_n} j_n^{j_n} k_n^{k_n} \sqrt{i_n}\sqrt{j_n}\sqrt{k_n}\; e^{-i_n - j_n - k_n}} \frac{1}{3^n}.$$

(By Stirling and $d=1$ calculation)

$$\le c' \frac{n^n}{3^n \left(\frac{n}{3}-1\right)^n \left(\frac{n}{3}-1\right)^{3/2}}$$

(use $i_n, j_n, k_n \ge \frac{n}{3} - 1$, $i_n + j_n + k_n = n$)

$$\le \frac{c''}{n^{3/2}} \left(\frac{n}{n-3}\right)^n \le \frac{c_0}{n^{3/2}}$$

$$\therefore \sum_{n=1}^{\infty} p_{\vec{0},\vec{0}}(2n) \le \sum_{n \ge 1}^{\infty} \frac{c_0}{n^{3/2}} < \infty$$

As $p_{\vec{0},\vec{0}}(2n+1) = 0$, this proves $\vec{0}$ is transient and so any state $i \in \mathbb{Z}^3$ is transient.