Math 302 Midterm 1

Instructor: Prof. Ed Perkins

Instructions:

• Write your name and student ID on every page.

• This examination contains four questions with total weight of 101 points.

• Write each answer very clearly below the corresponding question (Use back of page if needed). Simplify your answer as much as possible (but answers may be in terms of the exponential function, factorials, or “choose” symbols).

• No calculators, books, notebooks or any other written materials are allowed.

• Good luck!
1. (a) (10 pts) Define carefully: A probability $P$ on a given sample space $S$.

$P$ is a function from the subsets of $S$ to $\{0, 1\}$ such that

(i) $P(S) = 1$

(ii) If $\{A_n : n \in \mathbb{N}\}$ are disjoint subsets of $S$, then

$$P \left( \bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} P(A_n)$$

(b) (9 pts) Find the probability that a five card poker hand contains exactly 3 Aces.

$$P(\text{exactly 3 Aces}) = \left( \frac{4}{52} \right) \left( \frac{44}{51} \right) \left( \frac{42}{50} \right) \left( \frac{41}{49} \right)$$

Here $\left( \frac{4}{52} \right)$ = no. of ways to choose the 3 Aces

and $\left( \frac{44}{51} \right)$ = no. of ways to choose the 2 cards which are not Aces

(c) (10 pts) $A$ and $B$ are events such that $P(A) = .6$, $P(B) = .3$ and $P(A \cup B) = .8$. Find $P(A \cap B^c)$.

$A \cup B = (A \cap B^c) \cup B$

Disjoint

$P(A \cup B) = P(A \cap B^c) + P(B)$

$.8 = P(A \cap B^c) + .3$

$P(A \cap B^c) = .5$
There are 12 buses in the 100 Mile House bus fleet each with a capacity of 30 people.
Currently 6 of the buses are running full, 3 of them are running half full and 3 of them are running a sixth full.

(a) (6 pts) If a bus is chosen at random what is the probability that the bus is full?
\[ \frac{6}{12} = \frac{1}{2} \]

(b) (8 pts) If a bus rider is chosen at random what is the probability they are on a full bus?
\[
\text{Total no. of riders} = 6 \times 30 + 3 \times 15 + 3 \times 5 = 240 \\
\text{no. of riders on a full bus} = 6 \times 30 = 180 \\
P(\text{select a rider on a full bus}) = \frac{180}{240} = \frac{3}{4}
\]

(c) (10 pts) If a bus rider is chosen at random and Y is the number of people on the rider's bus, find \(E(Y)\). (You may leave your answer as a sum of explicit fractions.)
\[
Y = 5, 15 \; \text{or} \; 30. \\
P(Y = 30) = \frac{3}{4} \text{ by 16)} \\
\text{Similarly to 16: } P(Y = 15) = \frac{3 \times 15}{240} = \frac{45}{240} = \frac{3}{16} \\
P(Y = 5) = \frac{3 \times 5}{240} = \frac{15}{240} = \frac{1}{16} \\
E(Y) = 30 \cdot \frac{3}{4} + 15 \cdot \frac{3}{16} + 5 \cdot \frac{1}{16} \\
= \frac{90}{4} + \frac{45}{16} + \frac{5}{16} = \frac{215}{16} = 13.44 \\
= \frac{440}{16} = 25 \frac{5}{4}
\]
3. (a) (15 pts) The mean no. of misprints on a page of area 400 cm$^2$ is 10. A silver dollar of area 20 cm$^2$ is tossed on the page at random. Find the approximate probability that it covers at least one misprint.

\[ n = \frac{\text{no. of words on a page}}{\text{area}} = \frac{400}{20} = 20 \]

\[ p_n = 10/n \]

\[ p = \text{no. of words under the dollar} = \frac{20}{400} = \frac{n}{20} \]

\[ X = \text{no. of misprints under the dollar is binomial} \]

By Poisson approximation to binomial:

\[ P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-\lambda} \]

\[ \lambda = \frac{n}{20} = \frac{10}{2} = 5 \]

\[ = 1 - e^{-\frac{5}{2}} \]

(b) (15 pts) A fair die is tossed. If the outcome is $n$ then $n$ fair dice are tossed and $Y$ is the sum of these $n$ dice. If we are given that $Y = 3$ find the probability that $Y$ is the sum of 2 dice.

\[ X = \text{outcome of 1st die.} \]

\[ \text{Note that } Y = 3 \text{ implies } X = 1, 2 \text{ or } 3 \]

\[ P(X = 2 | Y = 3) = \frac{P(Y = 3 | X = 2) P(X = 2)}{P(Y = 3)} = \frac{3}{\sum_{n=1}^{3} P(Y = 3 | X = n) P(X = n)} \]

\[ = \frac{\frac{2}{36} \times \frac{1}{6}}{\left[ \frac{1}{6} + \frac{2}{36} + \frac{1}{36} \right] \frac{1}{6}} \]

\[ = \frac{12}{36 + 12} = \frac{12}{49} \]
4. $X_1$ is a binomial r.v. with parameters $n = 6, p = \frac{1}{3}$; $X_2$ is a geometric r.v. with parameter $p = \frac{1}{4}$, and $X_3$ is a Poisson r.v. with parameter $\lambda = 3$. Assume $X_1, X_2$ and $X_3$ are independent r.v.'s.

(a) (8 pts) Find $\text{Var}(X_1 + X_2 + X_3 + 1)$.

\[
\text{Var}(X_1 + X_2 + X_3) = \text{Var}(X_1) + \text{Var}(X_2) + \text{Var}(X_3) \\
= \omega \left( \frac{4}{3} \right) + \frac{1}{(1/4)^2} + 3 \\
= \frac{4}{3} + 16 + 3 \\
= 16 \frac{1}{3}
\]

(b) (10 pts) Show that $P(X_2 > X_1) = \left(\frac{11}{12}\right)^6$.

\[
P(X_2 > X_1) = \sum_{n=0}^{6} P(X_2 > n, X_1 = n) \\
= \sum_{n=0}^{6} P(X_2 > n) P(X_1 = n) \quad (\text{independence}) \\
= \sum_{n=0}^{6} \binom{6}{n} \left(\frac{1}{3}\right)^n \left(\frac{2}{3}\right)^{6-n} \left(\frac{1}{4}\right)^n \\
= \sum_{n=0}^{6} \binom{6}{n} \left(\frac{1}{4}\right)^n \left(\frac{2}{3}\right)^{6-n} \\
= \left(\frac{1}{4} + \frac{2}{3}\right)^6 \\
= \left(\frac{11}{12}\right)^6
\]