Nov. 7/11

Gammas, Exponentials, Chi-squared, Normals and Convolutions

Thm 2. Assume X is gamma(α,λ), Y is gamma(α',λ), and X and Y are independent. Then $X+Y$ is gamma$(\alpha+\alpha',\lambda)$.

Proof. By scaling it suffices to consider $\lambda=1$.
[Let $X = X'/\lambda$, $Y = Y'/\lambda$ where X' is gamma$(\alpha,1)$, Y' is gamma$(\alpha',1)$ and X', Y' are independent. If the result holds for $X'+Y'$, it will follow for $X+Y = (X'+Y')/\lambda$. Check this.]

Let $a>0$.

By Thm 1, $f_{X+Y}(a) = \int_{-\infty}^{\infty} f_Y(a-x)\, f_X(x)\, dx$

$$= \int_{-\infty}^{\infty} \frac{(a-x)^{\alpha-1} e^{-(a-x)}}{\Gamma(\alpha)} 1(a-x>0)\, \frac{x^{\alpha'-1} e^{-x}}{\Gamma(\alpha')} 1(x>0)\, dx$$

$$= \int_0^a \frac{(a-x)^{\alpha-1} x^{\alpha'-1}}{\Gamma(\alpha)\,\Gamma(\alpha')} e^{x} e^{-x}\, dx\; e^{-a}$$

$$= \frac{\int_0^a \left(1-\frac{x}{a}\right)^{\alpha-1} \left(\frac{x}{a}\right)^{\alpha'-1} \frac{dx}{a}\; a^{\alpha-1+\alpha'-1}\, a\, e^{-a}}{\Gamma(\alpha)\,\Gamma(\alpha')}$$

$$\left(w = \frac{x}{a},\ dw = \frac{dx}{a}\right) \quad = \underbrace{\left[\frac{\int_0^1 (1-w)^{\alpha-1} w^{\alpha'-1}\, dw}{\Gamma(\alpha)\,\Gamma(\alpha')} \times \Gamma(\alpha+\alpha')\right]}_{= C_{\alpha,\alpha'}} \underbrace{\frac{a^{\alpha+\alpha'-1} e^{-a}}{\Gamma(\alpha+\alpha')}}_{= f(a)}$$

So $f_{X+Y}(a) = C_{\alpha,\alpha'}\, f(a)$, where f is the density of gamma$(\alpha+\alpha', 1)$

But $1 = \int_{-\infty}^{\infty} f_{X+Y}(a)\, da = \int_{-\infty}^{\infty} C_{\alpha,\alpha'}\, f(a)\, da = C_{\alpha,\alpha'}$ so $f_{X+Y} = f$

i.e. $X+Y$ is gamma$(\alpha+\alpha', 1)$. $\square$

<u>Corollary 1</u> If $T_1, \ldots, T_n$ are independent exponential(λ) rv's, then $S_n = T_1 + T_2 + \ldots + T_n$ has a gamma(n, λ) distribution.

<u>Proof</u> $T_1, \ldots, T_n$ are independent gamma$(1, \lambda)$ rv's.
So by Thm 2, $T_1 + T_2$ is gamma$(2, \lambda)$
$(T_1 + T_2) + T_3$ is gamma$(3, \lambda)$
and so iterating we get $T_1 + \ldots + T_n$ is gamma(n, λ). $\square$

<u>Corollary 2</u> If $Z_1, \ldots, Z_n$ are independent standard normal rv's then $Y = Z_1^2 + Z_2^2 + \ldots + Z_n^2$ has a gamma$(\frac{n}{2}, \frac{1}{2})$ distribution,

i.e. $$f_Y(y) = \frac{\left(\frac{y}{2}\right)^{\frac{n}{2}-1} e^{-y/2} \left(\frac{1}{2}\right)}{\Gamma(n/2)} 1_{\{y>0\}}$$

<u>Remarks</u> ① Also say Y has a χ^2-distribution (chi-squared distribution) with n degrees of freedom.
② The χ^2-distribution arises in statistical estimation of μ and σ^2 for normal populations.

<u>Proof.</u> It suffices to show $(*)$ Z_1^2 is gamma$(\frac{1}{2}, \frac{1}{2})$.
Assume $(*)$. ~~By~~ Then $Z_1^2, \ldots, Z_n^2$ are independent gamma$(\frac{1}{2}, \frac{1}{2})$ rvs.
By Theorem 2, $Z_1^2 + \ldots + Z_n^2$ is gamma$(\frac{n}{2}, \frac{1}{2})$ and we would be done.
To prove $(*)$, recall we showed on Oct. 31 that

$$f_{Z_1^2}(x) = \frac{x^{-1/2} e^{-x/2}}{\sqrt{2\pi}} 1_{\{x>0\}}$$

The gamma$(\frac{1}{2}, \frac{1}{2})$ density is $$f(x) = \frac{\left(\frac{x}{2}\right)^{-\frac{1}{2}} e^{-x/2} \left(\frac{1}{2}\right)}{\Gamma(1/2)} 1_{\{x>0\}}$$

$$= \frac{x^{-1/2} e^{-x/2}}{\sqrt{2}\, \Gamma(1/2)} 1_{\{x>0\}}.$$

Since both $f_{2,2}$ and f must integrate to 1, it follows that $f_{2,2} = f$ (and $\Gamma(1/2) = \sqrt{\pi}$), and so (a) is proved. $\square$

<u>Thm 3.</u> If $X_1, X_2, \ldots, X_n$ are independent normal r.v.'s, and X_i is $N(\mu_i, \sigma_i^2)$, then $S_n = X_1 + X_2 + \ldots + X_n$ is a normal r.v. with $\mu = \sum_{i=1}^{n} \mu_i$, $\sigma^2 = \sum_{i=1}^{n} \sigma_i^2$.

<u>Proof.</u> We will give a proof based on moment-generating functions later. See p. 256 for a proof based on Theorem 1. To illustrate the idea we prove the result for $X_1 + X_2$ where X_1, X_2 are independent <u>standard</u> normals:

$$f_{X_1+X_2}(a) = \int_{-\infty}^{\infty} f_{X_2}(a-x) f_{X_1}(x)\, dx$$

(1) $$= \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}[(a-x)^2 + x^2]}}{\sqrt{2\pi}\,\sqrt{2\pi}}\, dx$$

Now $-\frac{1}{2}[(a-x)^2 + x^2] = -\frac{1}{2}[a^2 - 2ax + 2x^2] = -(x-\frac{a}{2})^2 - \frac{a^2}{4}$

$$\therefore f_{X_1+X_2}(a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(x-\frac{a}{2})^2 - a^2/4}\, dx$$

$$= \frac{e^{-a^2/2\cdot 2}}{2\pi}\int_{-\infty}^{\infty} e^{-w^2}\, dw \qquad w = x - \frac{a}{2}$$

$$w = \frac{z}{\sqrt{2}} \qquad dw = \frac{dz}{\sqrt{2}}$$

$$= \frac{e^{-a^2/2\cdot 2}}{2\pi}\int_{-\infty}^{\infty} e^{-z^2/2}\, \frac{dz}{\sqrt{2}}$$

$$= \frac{e^{-a^2/2\cdot 2}}{2\pi}\left(\sqrt{2\pi}\right)/\sqrt{2}$$

$$= \frac{e^{-a^2/2\cdot 2}}{\sqrt{2\pi}\,\sqrt{2}} \equiv \frac{e^{-a^2/2\sigma^2}}{\sqrt{2\pi}\,\sigma} \quad \text{where } \underline{\sigma^2 = 2}$$

$\therefore X_1 + X_2$ is $N(0, 2)$, as required.