Properties of Expectation (Nov. 16)

All r.v.'s $X, Y, ..$ are each discrete or continuous.

Theorem 1 (ES) (Expectation of Sums) $E(X+Y) = E(X) + E(Y)$
(Assuming all expectations are defined).

Proof. Let $X_n = X$ rounded off to n decimal places
$Y_n = Y$ " " " " " "

$X_n \in \{ \frac{k}{10^n} : k \in \mathbb{Z} \}$ is a discrete r.v., So is Y_n.

So, as we have proved the result in the discrete case, we know

$$(1) \quad E(X_n + Y_n) = E(X_n) + E(Y_n).$$

We will prove the result by letting $n \to \infty$ in (1).
To do this recall from Nov. 9 lecture:

(2) Z, W rvs such that $Z \le W \Rightarrow E(Z) \le E(W)$

By definition $X - 10^{-n} \le X_n \le X + 10^{-n}$, So (2) implies

(3) $E(X) - 10^{-n} \le E(X - 10^{-n}) \le E(X_n) \le E(X + 10^{-n}) = E(X) + 10^{-n}$
(we showed $E(aX+b) = aE(X) + b$ earlier).

(3) implies $\lim_{n\to\infty} E(X_n) = E(X)$. Similarly one shows $\lim_{n\to\infty} E(Y_n) = E(Y)$ and

$\lim_{n\to\infty} E(X_n+Y_n) = E(X+Y)$. So we may let $n \to \infty$ in (1) to conclude

$$E(X+Y) = E(X) + E(Y). \quad \square$$

Theorem 2. (Product Rule) X, Y independent r.vs $\Rightarrow E(XY) = E(X)E(Y)$
(Assuming all expectations exist).

Proof. Let X_n, Y_n be as in the proof of Theorem 1.
So by the (already proved) product rule for discrete r.vs we have

(1) $\quad E(X_n Y_n) = E(X_n)E(Y_n)$.

(Since $X_n = g_n(X)$ and $Y_n = h_n(Y)$ are independent discrete r.v.s)

Our goal is to prove the result by letting $n \to \infty$ in (1).
We know from the proof of Thm 1 that

(2) $\quad \lim\limits_{n\to\infty} E(X_n)E(Y_n) = \left(\lim\limits_{n\to\infty} E(X_n)\right)\left(\lim\limits_{n\to\infty} E(Y_n)\right) = E(X)E(Y)$.
(arithmetic of limits)

We have

$$
\begin{aligned}
|X_n Y_n - X\cdot Y| &\le |X_n \cdot Y_n - X\cdot Y_n| + |X\cdot Y_n - X\cdot Y| \\
&\le |X_n - X||Y_n| + |X||Y_n - Y| \\
&\le 10^{-n}(|Y| + 10^{-n}) + |X|\,10^{-n} \qquad \because |Y_n| \le |Y| + 10^{-n} \\
&= 10^{-n}|Y| + 10^{-n}|X| + 10^{-2n}
\end{aligned}
$$

So $\quad XY - 10^{-n}|X| - 10^{-n}|Y| - 10^{-2n} \le X_n Y_n \le XY + 10^{-n}|X| + 10^{-n}|Y| + 10^{-2n}$

$$
\therefore E\left(XY - 10^{-n}|X| - 10^{-n}|Y| - 10^{-2n}\right) \le E(X_n Y_n) \le E\left(XY + 10^{-n}|X| + 10^{-n}|Y| + 10^{-2n}\right)
$$

(by Thm 1)

$E(XY) - 10^{-n}[E(|X|) + E(|Y|)] - 10^{-2n} \qquad\qquad E(XY) + 10^{-n}(E(|X|) + E(|Y|)) + 10^{-2n}$

$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$

$E(XY)$ as $n\to\infty$ $\qquad\qquad\qquad\qquad$ $E(XY)$ as $n \to \infty$

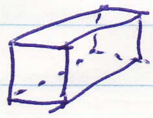
By Squeeze Thm: (3) $\lim E(X_n Y_n) = E(XY)$.
By (2) and (3) we can let $n \to \infty$ in (1) to conclude

$$E(XY) = E(X)E(Y).$$

Definition Random variables $X_1, \ldots, X_n$ have **joint** p.d.f. $f(x_1, x_2, \ldots, x_n) \geq 0$

iff $\quad P((X_1, X_2, \ldots, X_n) \in C) = \int_C f(x_1, \ldots, x_n)\, \underbrace{d\underline{x}}_{=\, dx_1 dx_2 \ldots dx_n}$ $\quad$ for "all" sets C in $\mathbb{R}^n$.

Note: It suffices to verify the above for all "boxes" $C = I_1 \times I_2 \times \ldots \times I_n$, I_j intervals.



Theorem (LUS II) Assume $X_1, \ldots, X_n$ have joint p.d.f. f and

$g : \mathbb{R}^n \to \mathbb{R}$. Then $\quad E(g(X_1, \ldots, X_n)) = \int_{\mathbb{R}^n} g(x_1, \ldots, x_n) f(x_1, \ldots, x_n)\, d\underline{x}$

(providing $\int_{\mathbb{R}^n} |g(x_1, \ldots, x_n)| f(x_1, \ldots, x_n)\, d\underline{x} < \infty$).

Proof. Same as that for $n=1$ which was given earlier as (LUS II).
[Check out the $n=2$ proof on p249, for example]. $\square$