Properties of Expectations (Nov. 16)

All r.v.'s $X_1, \ldots, X_n$ are each discrete or continuous.

**Theorem (ES)** (Expectation of Sum)

$E(X + Y) = E(X) + E(Y)

(assuming all expectations are defined)

**Proof:** Let $X_n = X$ rounded off to $n$ decimal places

$Y_n = Y$ rounded off to $n$ decimal places

$X_n \rightarrow X$ as $n \rightarrow \infty$ is a discrete r.v.

$Y_n \rightarrow Y$ as $n \rightarrow \infty$ is a discrete r.v.

So, as we have proved the result in the discrete case, we know

(1) $E(X_n + Y_n) = E(X_n) + E(Y_n)$

We will prove the result by letting $n \rightarrow \infty$ in (1).

To do this recall from Nov. 9 lecture:

(2) $Z, W$ r.v's such that $Z \leq W \Rightarrow E(Z) \leq E(W)$

By definition $X - 10^{-n} \leq X_n \leq X + 10^{-n}$

So (2) implies

(3) $E(X) - 10^{-n} \leq E(X_n) \leq E(X) + 10^{-n} \leq E(X + 10^{-n}) = E(X) + 10^{-n}

(we showed $E(U + V) = E(U) + E(V)$ earlier)

(3) implies $\lim_{n \rightarrow \infty} E(X_n) = E(X)$. Similarly one shows $\lim_{n \rightarrow \infty} E(Y_n) = E(Y)$ and

$\lim_{n \rightarrow \infty} E(X_n Y_n) = E(XY)$.

So we may let $n \rightarrow \infty$ to conclude

$E(X + Y) = E(X) + E(Y)$. \n
Theorem 2. (Product Rule) \( X, Y \) independent r.v.s. \( \Rightarrow E(XY) = E(X)E(Y) \)

Proof: Let \( X_n, Y_n \) be as in the proof of Theorem 1. So by the already proved product rule for discrete r.v.s we have

1) \[ E(X_nY_n) = E(X_n)E(Y_n). \]

(Since \( X_n = g_n(X) \) and \( Y_n = h_n(Y) \) are independent discrete r.v.s.)

Our goal is to prove the result by letting \( n \to \infty \) in (1).

We know from the proof of Theorem 1 that

2) \[ \lim_{n \to \infty} E(X_n)E(Y_n) = \left( \lim_{n \to \infty} E(X_n) \right) \left( \lim_{n \to \infty} E(Y_n) \right) = E(X)E(Y). \]

We have

\[ |X_nY_n - XY| \leq |X_nY_n - X \cdot Y_n| + |X \cdot Y_n - X \cdot Y| \]

\[ = |X_n - X| |Y_n| + |X||Y_n - Y| \]

\[ \leq 10^n |Y| + 10^n |Y| \]

\[ = 10^{2n}|Y| + 10^n |Y| + 10^{-n} \]

\[ \leq 10^{2n}|Y| + 10^{-n} |X| + 10^{-n} \]

\[ \leq 10^{2n}|Y| + 10^{-n} |X| + 10^{-n} \]

So

\[ \lim_{n \to \infty} E(X_nY_n) = E(XY) \]

(3) \[ \lim_{n \to \infty} E(X_nY_n) = E(XY). \]

By Squeeze Thm: (3) \[ \lim_{n \to \infty} E(X_nY_n) = E(XY). \]

By (2) and (3) we can let \( n \to \infty \) in (1) to conclude

\[ E(XY) = E(X)E(Y). \]
Definition: Random variables $X_1, \ldots, X_n$ have joint p.d.f. $f(x_1, x_2, \ldots, x_n) > 0$ if

$$P((X_1, X_2, \ldots, X_n) \in C) = \frac{\int_{x_1 \in I_1} \int_{x_2 \in I_2} \cdots \int_{x_n \in I_n} f(x_1, x_2, \ldots, x_n) \, dx_1 \, dx_2 \cdots dx_n}{\int_{x_1 \in I_1} \int_{x_2 \in I_2} \cdots \int_{x_n \in I_n} 1 \, dx_1 \, dx_2 \cdots dx_n}$$

Note: It suffices to verify the above for all "boxes" $C = I_1 \times I_2 \times \cdots \times I_n$, $I_j$ intervals.

Theorem (LU5.3) Assume $X_1, \ldots, X_n$ have joint p.d.f. $f$ and $g: \mathbb{R}^n \to \mathbb{R}$. Then $E(g(X_1, \ldots, X_n)) = \int_{\mathbb{R}^n} g(x_1, \ldots, x_n) f(x_1, \ldots, x_n) \, dx$

(providing $\int_{\mathbb{R}^n} |g(x_1, \ldots, x_n)| f(x_1, \ldots, x_n) \, dx < \infty$).

Proof. Same as that for $n=1$ which was given earlier as (LU5.2). I check out the $n=2$ proof on p.248, for example $\square$