We now consider the problem of constructing probability measures on \( \prod_{t \in T} \mathcal{F}_t \). The approach will be as follows: Let \( v = \{t_1, \ldots, t_n\} \) be a finite subset of \( T \), where \( t_1 < t_2 < \cdots < t_n \). (If \( T \) is not a subset of \( R \), some fixed total ordering is put on \( T \).) Assume that for each such \( v \) we are given a probability measure \( P_v \) on \( \prod_{t \in T} \Omega_t \); \( P_v(B) \) is to represent \( P(\omega \in \prod_{t \in T} \Omega_t : (\omega(t_1), \ldots, \omega(t_n)) \in B) \). We shall require that the \( P_v \) be “consistent”; to see what kind of consistency is needed, consider an example.

Suppose \( \mathcal{T} \) is the set of positive integers and \( \mathcal{F}_t = g(R) \) for all \( t \). Suppose we know \( P_{\{v\}}(B_{\{v\}}) = P(\omega : (\omega(t_1), \ldots, \omega(t_n)) \in B_{\{v\}}) \) for all \( \omega(t_1), \ldots, \omega(t_n) \in B_{\{v\}} \). Then \( P(\omega : (\omega(t_1), \omega(t_2), \omega(t_3)) \in B_{\{v\}}) \) for all \( \omega(t_1), \omega(t_2), \omega(t_3) \in B_{\{v\}} \). Thus once probabilities of sets involving the first five coordinates are specified, probabilities of sets involving \( \omega(t_1) \) [as well as \( \omega(t_2), \omega(t_3), \omega(t_4) \), and so on], are determined. Thus the original specification of \( P_{\{v\}} \) must agree with the measure induced from \( P_{\{v\}} \).

We are now ready to formalize: If \( v = \{t_1, \ldots, t_n\} \) be a finite subset of \( T \), where \( t_1 < t_2 < \cdots < t_n \), the space \( \{\omega \in \Omega : \omega(t_1), \ldots, \omega(t_n) \in B_{\{v\}}\} \) is denoted by \( \Omega(v) \). If \( u = \{t_1, \ldots, t_k\} \) is a nonempty subset of \( v \) and \( y = (y(t_1), \ldots, y(t_k)) \in \Omega_v \), the \( k \)-tuple \( (y(t_1), \ldots, y(t_k)) \) is denoted by \( y_u \). Similarly if \( \omega = (\omega(t), t \in T) \) belongs to \( \prod_{t \in T} \Omega_t \), the notation \( \omega_u \) will be used for \( (\omega(t_1), \ldots, \omega(t_k)) \).

If \( P_v \) is a probability measure on \( \mathcal{F}_v \), the projection of \( P_v \) on \( \mathcal{F}_u \) is the probability measure \( \pi_u(P_v) \) on \( \mathcal{F}_u \) defined by

\[
[\pi_u(P_v)](B) = P_v(\omega \in \Omega_v : \omega_u \in B), \quad B \in \mathcal{F}_v.
\]

Similarly, if \( Q \) is a probability measure on \( \prod_{t \in T} \mathcal{F}_t \), the projection of \( Q \) on \( \mathcal{F}_v \) is defined by

\[
[\pi_v(Q)](B) = Q(\omega \in \prod_{t \in T} \Omega_t : \omega_u \in B) = Q(B(v)), \quad B \in \mathcal{F}_v.
\]

We need one preliminary result.

### 4.4.2 Theorem
For each \( n = 1, 2, \ldots \), suppose that \( \mathcal{F}_n \) is the class of Borel sets of a separable metric space \( \Omega_n \). Let \( \Omega = \prod_n \Omega_n \), with the product topology, and let \( \mathcal{F} = g(\Omega) \). [Note that \( \Omega \) is metrizable, so that the Baire and Borel sets of \( \Omega \) coincide; we may take

\[
d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x_n, y_n)},
\]

where \( d_n \) is the metric of \( \Omega_n \). Also, if each \( \Omega_n \) is complete, so is \( \Omega \).]

Then \( \mathcal{F} \) is the product \( \sigma \)-field \( \prod_n \mathcal{F}_n \).

**Proof.** The sets \( \{\omega \in \Omega : \omega_1 \in A_1, \ldots, \omega_n \in A_n\}, n = 1, 2, \ldots \), where the \( A_i \) range over the countable base for \( \Omega_i \) (recall that separability and second countability are equivalent in metric spaces), form a countable base for \( \Omega \). Since the sets are measurable rectangles, it follows that every open subset of \( \Omega \) belongs to \( \prod_n \mathcal{F}_n \); hence \( \mathcal{F} \subseteq \prod_n \mathcal{F}_n \). On the other hand, for a fixed positive integer \( i \) let \( \mathcal{B} = \{B \in g(\Omega_i) : (\omega \in \Omega : \omega_i \in B) \in \mathcal{F}\} \). Then \( \mathcal{B} \) is a \( \sigma \)-field containing the open sets of \( \Omega_i \), hence \( \mathcal{B} = g(\Omega_i) \). In other words, every measurable rectangle with one-dimensional base belongs to \( \mathcal{F} \). Since an arbitrary measurable rectangle is a finite intersection of such sets, it follows that \( \prod_n \mathcal{F}_n \subseteq \mathcal{F} \).

We are now ready for the main result.

### 4.4.3 Kolmogorov Extension Theorem
For each \( i \) in the arbitrary index set \( T \), let \( \Omega_i \) be a complete, separable metric space, and \( \mathcal{F}_i \) the Borel sets of \( \Omega_i \). Assume that for each finite nonempty subset \( v \) of \( T \), we are given a probability measure \( P_v \) on \( \mathcal{F}_v \). Assume the \( P_v \) are consistent, that is, \( \pi_u(P_v) = P_u \) for each nonempty \( u \subset v \).

Then there is a unique probability measure \( P \) on \( \mathcal{F} = \prod_{t \in T} \mathcal{F}_t \), such that \( \pi_t(P) = P_t \) for all \( v \).

**Proof.** We define the hoped-for measure on measurable cylinders by

\[
P(B^v) = \pi_v(B^v), \quad B^v \in \mathcal{F}_v.
\]
We must show that this definition makes sense since a given measurable cylinder can be represented in several ways. For example, if all $\Omega_i = \mathbb{R}$ and $B^3 = (-\infty, 3) \times (4, 5) \times \mathbb{R}$, then

$$B^2(t_1, t_2) = \{\omega: \omega(t_1) < 3, \ 4 < \omega(t_2) < 5\}$$

$$= \{\omega: \omega(t_1) < 3, \ 4 < \omega(t_2) < 5, \ \omega(t_3) \in \mathbb{R}\}$$

$$= B^3(t_1, t_2, t_3) \text{ where } B^3 = (-\infty, 3) \times (4, 5) \times \mathbb{R}.$$  

It is sufficient to consider dual representation of the same measurable cylinder in the form $B^w(u) = B^k(u)$ where $k < n$ and $u < v$. But then

$$P_w(B^k) = \left[\pi_v(P_u)(B^k)\right] \text{ by the consistency hypothesis}$$

$$= P_v(y \in \Omega_v; \ y_u \in B^k) \text{ by definition of projection}. $$

But the assumption $B^w(u) = B^k(u)$ implies that if $y \in \Omega_v$, then $y \in B^v$ if $y_u \in B^k$, hence $P_w(B^k) = P_v(B^k)$, as desired.

Thus, $P$ is well-defined on measurable cylinders; the class $\mathcal{F}_0$ of measurable cylinders forms a field, and $\sigma(\mathcal{F}_0) = \mathcal{F}$.

Now if $A_1, \ldots, A_m$ are disjoint sets in $\mathcal{F}_0$, we may write (by introducing extra factors as in the above example) $A_i = B^w(v_i), i = 1, \ldots, m$, where $v = \{t_1, \ldots, t_k\}$ is fixed and the $B^w$, $i = 1, \ldots, m$, are disjoint sets in $\mathcal{F}_v$. Thus

$$P\left(\bigcup_{i=1}^m A_i\right) = P\left(\bigcup_{i=1}^m B^w(v_i)\right)$$

$$= P_v\left(\bigcup_{i=1}^m B^w\right) \text{ by definition of } P$$

$$= \sum_{i=1}^m P_v(B^w) \text{ since } P_v \text{ is a measure}$$

$$= \sum_{i=1}^m P(A_i) \text{ again by definition of } P.$$ 

Therefore $P$ is finitely additive on $\mathcal{F}_0$. To show that $P$ is countably additive on $\mathcal{F}_0$, we must verify that $P$ is continuous from above at $\emptyset$ and invoke 1.2.8(b). The Carathéodory extension theorem (1.3.10) then extends $P$ to $\mathcal{F}$.

Let $A_k, k = 1, 2, \ldots$ be a sequence of measurable cylinders decreasing to $\emptyset$. If $P(A_k)$ does not approach 0, we have, for some $\varepsilon > 0$, $P(A_k) \geq \varepsilon > 0$ for all $k$. Suppose $A_k = B^w(v_k)$; by tacking on extra factors, we may assume that the numbers $n_k$ and the sets $v_k$ increase with $k$.

By 4.4.2, each $\Omega_{v_k}$ is a complete, separable metric space and $\mathcal{F}_{v_k} = \mathcal{B}(\Omega_{v_k})$. It follows from 4.3.8 that we can find a compact set $C_n \subset B^w$ such that $P_{v_k}(B^w - C_n) < 2^{-k+1}$. Define $A_k' = C_n(v_k) \subset A_k$. Then $P(A_k - A_k') = P(A_k) - P(A_k') < 2^{-k+1}$. In this way we approximate the given cylinders by cylinders with compact bases.

Now take $D_k = A_1' \cap A_2' \cap \cdots \cap A_k' \subset A_1 \cap A_2 \cap \cdots \cap A_k = A_k$. Then

$$P(A_k - D_k) = P\left(A_k \cap \bigcap_{i=1}^k A_i'\right) \leq \sum_{i=1}^k P(A_{i_k} \cap A_i')$$

$$\leq \sum_{i=1}^k P(A_{i_k} - A') \leq \sum_{i=1}^k \varepsilon/2^{k+1} < \varepsilon/2.$$ 

Since $D_k \subset A_k' \subset A_k$, $P(A_k - D_k) = P(A_k) - P(D_k)$, consequently $P(D_k) > P(A_k) - \varepsilon/2$. In particular, $D_k$ is not empty.

Now pick $x_k \in D_k, k = 1, 2, \ldots$. Say $A_1' = C_n(t_1, \ldots, t_n) = C_n(r_1)$ (note all $D_k \subset A_1'$). Consider the sequence

$$(x_1^1, \ldots, x_1^n), \quad (x_2^1, \ldots, x_2^n), \quad (x_3^1, \ldots, x_3^n), \quad \ldots,$$

that is, $x_1^1, x_2^2, x_3^3, \ldots$.

Since the $x_n$ belong to $C_n$, a compact subset of $\Omega_{v_n}$, we have a convergent subsequence $x_n^m$ approaching some $x_0 \in C_n$. If $A_2' = C_n'(t_2, \ldots, t_n)$ (so $D_k \subset A_2'$ for $k \geq 2$), consider the sequence $x_2^1, x_2^2, x_2^3, \ldots \in C_n'$ (eventually), and extract a convergent subsequence $x_2^m \to x_2 \in C_n'$. Note that $(x_2^m)^m \to x_2$, as $n \to \infty$, the left side approaches $(x_2^m)^m$, and since $\{r_{2n}\}$ is a subsequence of $\{r_1\}$, the right side approaches $x_0$. Hence $(x_0, i_0, e_0) = x_0$.

Continue in this fashion; at step $i$ we have a subsequence

$$x_{i_1'} \to x_0 \in C_n,$$

and $(x_{i_1}^j)_{i_1} = x_{i_1}^j$ for $j < i$.

Pick any $\omega \in \bigcap_{i=1}^{t} \Omega_i$, such that $\omega_{j_1} = x_{i_1}^j$ for all $j_1 = 1, 2, \ldots$ (such a choice is possible since $(x_{i_1}^j)_{i_1} = x_{i_1}^j$, $j < i$). Then $\omega_{j_1} \in C_{n'}$ for each $j$; hence

$$\omega \in \bigcap_{j=1}^{\infty} A_j' \subset \bigcap_{j=1}^{\infty} A_j = \emptyset,$$

a contradiction. Thus $P$ extends to a measure on $\mathcal{F}$, and by construction, $\pi_v(P) = P_v$ for all $v$.

Finally, if $P$ and $Q$ are two probability measures on $\mathcal{F}$ such that $\pi_v(P) = \pi_v(Q)$ for all finite $v \subset T$, then for any $B^w \in \mathcal{F}_v$,

$$P(B^w(v)) = [\pi_v(P)](B^w) = [\pi_v(Q)](B^w) = Q(B^w(v)).$$

Thus $P$ and $Q$ agree on measurable cylinders, and hence on $\mathcal{F}$ by the uniqueness part of the Carathéodory extension theorem.