Solutions to Homework 9

1. $P(|\bar{X}_n-\mu| \le .1) = P\left(\frac{|\bar{X}_n-\mu|}{\sigma/\sqrt{n}} \le \frac{.1\sqrt{n}}{\sigma}\right)$

$\approx P\left(|Z| \le \frac{.1\sqrt{n}}{\sigma}\right)$ $\quad Z \sim N(0,1)$ n large. by CLT

So want $P\left(|Z| \le \frac{.1\sqrt{n}}{\sigma}\right) \ge .95$

which will be true if $\left(\frac{.1\sqrt{n}}{\sigma}\right) \ge z_{.025} = 1.96$

$\iff n \ge \left(\frac{1.96 \times \sigma}{.1}\right)^2 = 100\,\sigma^2 (1.96)^2$

$\Uparrow$

$(\because \sigma^2 \le 10)$. $\quad n \ge 1000\,(1.96)^2 = 3841.6$.

So $n \ge 3842$ will do.

Using Chebychev required $n \ge 20{,}000$.

2. Let X_i = lifetime of i^{th} lamp. $X_1 \ldots X_{25}$ are iid $\mu = 50$, $\sigma = 4$.

$P\left(\sum_{i=1}^{25} X_i > 1300\right) = P\left(\frac{\sum_{i=1}^{25} X_i - 25\times 50}{4\sqrt{25}} > \frac{1300-(25\times 50)}{4\sqrt{25}}\right)$

(CLT) $\quad \approx P\left(Z \ge \frac{50}{20}\right) = P(Z \ge 2.5) = 1-\Phi(2.5) = .0062$.

[But $n = 25$ is not that large so error could be significant)

3. $Var(Y_\varepsilon) = Var(bX + \varepsilon Z) = b^2 Var(X) + \varepsilon^2 Var(Z) = b^2 + \varepsilon^2$.

$Cov(X, Y_\varepsilon) = Cov(a + bX + \varepsilon Z, X) = b\,Cov(X,X) + \varepsilon\, \underset{0}{Cov(Z,X)} = b\,Var(X) = b$

$\therefore \rho(X, Y_\varepsilon) = \frac{Cov(X,Y_\varepsilon)}{\sigma_X \sigma_{Y_\varepsilon}} = \frac{b}{1\cdot\sqrt{b^2+\varepsilon^2}} = \frac{b}{\sqrt{b^2+\varepsilon^2}}$

$\therefore \lim_{\varepsilon\to 0^+} \rho(X, Y_\varepsilon) = \begin{cases} +1 & b>0 \\ -1 & b<0 \end{cases}$ (ε small $\Rightarrow$ strong linear dependence between X and Y_ε)

$\lim_{\varepsilon\to\infty} \rho(X, Y_\varepsilon) = 0$ [ε large, linear dependence is swamped by the "noise" εZ).

4. $X_i = \mu + \sigma Z_i$ $Z_i = \frac{X_i - \mu}{\sigma}$ iid $N(0,1)$

$\therefore \bar{X}_n = \sum_{i=1}^{n} \frac{\mu + \sigma Z_i}{n} = \mu + \sum_{i=1}^{n} \frac{\sigma Z_i}{n}$

$\therefore \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} = \sum_{i=1}^{n} \frac{\sigma Z_i/n}{\sigma/\sqrt{n}} = \sum_{i=1}^{n} \frac{Z_i}{\sqrt{n}}$ is $N(0, \sigma^2)$ ~~where~~ $\sigma^2 = \sum_{i=1}^{n} \frac{1}{n} = 1$.

$\therefore \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}$ is $N(0,1)$.

So apply the CLT with X_i $N(\mu, \sigma^2)$ where the limit is denoted Z. $\hat{X}_n$ is $N(0,1)$.

$\therefore F_Z(x) = \lim_{n\to\infty} F_{\hat{X}_n}(x) = \Phi(x) =$ cdf of $N(0,1)$

5. Let X_i = net winnings in i^{th} play. Y_i = result of i^{th} toss $\in \{1,2,3,4,5,6\}$.

$X_i = 10Y_i - 40$ $E(X_i) = 10E(Y_i) - 40 = 10 \times 3.5 - 40 = \underline{-5}$.

$Var(Y_i) = \sum_{j=1}^{6} j^2 \frac{1}{6} - E(Y_i)^2 = \frac{1+4+9+16+25+36}{6} - \left(\frac{7}{2}\right)^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$

$\therefore Var(X_i) = 100\, Var(Y_i) = \frac{3500}{12}$ $\therefore \sigma_{X_i} = 10\sqrt{\frac{35}{12}}$

$X_1 \dots X_{30}$ iid so by CLT.

$P\left(\sum_{i=1}^{30} X_i > 0\right) = P\left(\frac{\sum_{i=1}^{30} X_i - 30(-5)}{\sqrt{30} \times 10\sqrt{\frac{35}{12}}} > \frac{150}{\sqrt{30} \times 10\sqrt{\frac{35}{12}}}\right)$

$\approx P\left(Z > \frac{30}{\sqrt{350}}\right) \approx 1 - \Phi(1.60) \approx 1 - .9452 \approx \underline{.055}$

6. (a) $P(\text{none has birthday on Jan. 1}) = \prod_{i=1}^{20} \left(1 - 1/365\right) = \left(\frac{364}{365}\right)^{20}$ (ind'ce)

$\therefore P(\text{at least 1 person has birthday on Jan. 1}) = 1 - \left(\frac{364}{365}\right)^{20}$

(b) N = # of distinct birthdays in 20 people $= \sum_{j=1}^{365} X_j$.

$E(X_j) = P(\text{at least 1 person has birthday on day } j) = 1 - \left(\frac{364}{365}\right)^{20}$.

$\therefore E(N) = \sum_{j=1}^{365} E(X_j) = 365\left[1 - \left(\frac{364}{365}\right)^{20}\right]$. □