Solutions to Homework 6.

1. (a) X = ht. of randomly chosen Cdn female over 21

X is $N(64, 9)$ $\therefore$ $Z = (X-64)/3$ is standard normal

$$P(X > 70) = P\left(\frac{X-64}{3} > \frac{70-64}{3}\right) = P(Z > 2) = 1 - P(Z \le 2)$$

2 $$\doteq 1 - .9772$$

$$= .0228$$

1(b) $$P(X > 72) = P\left(Z > \frac{72-64}{3}\right) = P\left(Z > 2\tfrac{2}{3}\right) = 1 - P\left(Z \le 2\tfrac{2}{3}\right)$$

$$\doteq 1 - .9962$$

$$= .0038$$

2 So $$P(X > 72 \mid X > 70) = \frac{P(X>72)}{P(X>70)} \doteq \frac{.0038}{.0228} \doteq .167$$

So about ~~1.67%~~ 16.7% of the women over 5'10" are over 6'.

2. $$P(\min(X_1, X_2) > x) = P(X_1 > x)\,P(X_2 > x) = e^{-\lambda_1 x} e^{-\lambda_2 x}$$

$$= e^{-(\lambda_1+\lambda_2)x}$$

2 $\therefore$ $P(\min(X_1, X_2) \le x) = 1 - e^{-(\lambda_1+\lambda_2)x}$

$= P(X \le x)$ where X is exponential $(\lambda_1+\lambda_2)$.

$\therefore$ $\min(X_1, X_2)$ is exponential $(\lambda_1+\lambda_2)$ since it has the cdf of an exponential $(\lambda_1+\lambda_2)$.

3. $X_1 =$ (time of arrival of 329 in hours) $- 5$ is unif. on $[0,1]$

independent

$X_2 =$ (time of arrival of 330 in hours) $- 5$ is unif. on $[0, \frac{1}{2}]$

(a) $$P(\text{on bus by 5:10}) = 1 - P\left(X_1 > \tfrac{10}{60}, X_2 > \tfrac{10}{60}\right)$$

$$= 1 - P\left(X_1 > \tfrac{10}{60}\right) P\left(X_2 > \tfrac{10}{60}\right)$$

2

$$P(\text{on bus by } 5{:}10) = 1 - \left(\int_{1/6}^{1} dx\right)\left(\int_{1/6}^{1/2} 2\,dy\right)$$

2

$$= 1 - \frac{5}{6}\cdot\frac{2}{3} = 1 - 5/9 = \boxed{\frac{4}{9}}.$$

1b) $P\left(X_1 > \frac{20}{60}, X_2 > \frac{20}{60} \,\middle|\, X_1 > \frac{10}{60}, X_2 > \frac{10}{60}\right)$

$$= P\left(X_1 > \frac{20}{60}\right) P\left(X_2 > \frac{20}{60}\right) \Big/ 5/9 \qquad \left(\because P\left(X_1 > \frac{10}{60}, X_2 > \frac{10}{60}\right) = 1 - \frac{4}{9} = \frac{5}{9}\right).$$

$$= \left(\int_{1/3}^{1} dx\right)\left(\int_{1/3}^{1/2} 2\,dx\right) \Big/ 5/9$$

$$= 2/3 \cdot 1/3 \Big/ 5/9 \quad = \boxed{2/5}.$$

4. If $x>0$, $P(X/\lambda \le x) = P(X \le \lambda x) = \int_0^{\lambda x} e^{-w}\,dw \qquad w = \lambda u \quad dw = \lambda\,du$

$$= \int_0^{x} e^{-u\lambda}\,\lambda\,du.$$

2

$\therefore$ $\cancel{\ldots} = P(Y \le x) \qquad Y$ exponential(λ)

$\therefore X/\lambda$ is an exponential(λ) r.v. (have the same c.d.f.)

5. $$E(Z^{n+1}) = \int_{-\infty}^{\infty} z^{n+1} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz = \int_{-\infty}^{\infty} \underbrace{z^n}_{f} \underbrace{\left(z e^{-z^2/2}\right) dz}_{dg} \frac{1}{\sqrt{2\pi}}$$

$$= \frac{z^n\left(-e^{-z^2/2}\right)}{\sqrt{2\pi}}\Bigg|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} n z^{n-1} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz$$

3

$$= 0 + n\,E(Z^{n-1})$$

Take $n=3$: $E(Z^4) = 3\,E(Z^2) = 3\,\mathrm{Var}(Z) = 3$.

6. (a) $P(X=0) = \Delta F_X(0) = \frac{1}{2} - 0 = \frac{1}{2}$

1

1 (b) $P(\frac{1}{4} < X < \frac{1}{2}) = F_X(\frac{1}{2}-) - F_X(\frac{1}{4}) = \frac{3}{4} - (\frac{1}{2} + (\frac{1}{4})^2)$

lefthand limit at $\frac{1}{2}$ $= \frac{12}{16} - \frac{9}{16} = \frac{3}{16}$

1 (c) $P(\frac{3}{4} < X \le 1) = F_X(1) - F_X(3/4)$

$= 1 - \frac{3}{4} = 1/4$

(d) $E(X) = \int_0^\infty P(X > x)\,dx$ since $\underline{X \ge 0}$

$= \int_0^\infty (1 - F_X(x))\,dx$

3 $= \int_0^{1/2} 1 - (\frac{1}{2} + x^2)\,dx + \int_{1/2}^1 1 - 3/4\,dx$

$= \frac{1}{4} - \int_0^{1/2} x^2\,dx + \frac{1}{8}$ ~~[illegible]~~

$= \frac{3}{8} - \frac{1}{3}\cdot(\frac{1}{2})^3 = \frac{1}{3}$

(e) If X was discrete, $\sum_x \Delta F_X(x) = \sum_x p(x)$ would equal 1.

But $\Delta F_X(0) = \frac{1}{2}$ $\Delta F_X(1) = 1/4$ and $\Delta F_X(x) = 0$ for $x \ne 0$ or 1

$\therefore \sum_x \Delta F_X(x) = \frac{1}{2} + \frac{1}{4} = 3/4 < 1$.

4 $\therefore X$ is $\underline{\text{not}}$ a discrete r.v.

If X was cont's, $F_X(x) = \int_{-\infty}^x f_X(w)\,dw$ would be continuous

in x. But F_X has jumps at $x=0$ and $x=1$.

$\therefore X$ is $\underline{\text{not}}$ a continuous r.v.

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