Solutions to Assignment 5

1 (a) $$P(0<X<2) = \int_1^2 x^{-2}\,dx = 1 - \tfrac{1}{2} = \tfrac{1}{2}$$

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1 (b) $$P(0<Y<2) = \int_0^2 7e^{-7x}\,dx = \int_0^{14} e^{-u}\,du = 1 - e^{-14}$$

2. (a) $$P(X\le a) = \tfrac{1}{2} \iff \int_0^a \tfrac{1}{8}x\,dx = \tfrac{1}{2} \qquad (a<4)$$

$$\iff \tfrac{1}{16}a^2 = \tfrac{1}{2} \iff \underline{a=\sqrt{8}}$$

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(b) $$P(X\ge a) = \tfrac{1}{4} \iff \int_a^4 \tfrac{1}{8}x\,dx = \tfrac{1}{4}$$

$$\iff \tfrac{1}{16}\left[16-a^2\right] = \tfrac{1}{4} \iff \tfrac{a^2}{16} = \tfrac{3}{4} \iff a = \sqrt{12}$$

3. $$\int_0^a c x^4\,dx = 1 \quad \text{for } f \text{ to be a density}$$

$$\iff (1)\ c a^5 = 5$$

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$$1 = E(X) \iff \int_0^a c x^5\,dx = 1 \iff (2)\ c a^6 = 6 \iff (c a^5)a = 6$$

So by (1) $5a = 6$ (3) $\underline{\underline{a = 6/5}}$

$$E(X^2) = \int_0^a c x^6\,dx = \frac{ca^7}{7} = \frac{(ca^6)a}{7} = \frac{6}{7}\cdot\frac{6}{5} \qquad \text{by (2) and (3)}$$

$$\therefore \operatorname{Var}(X) = E(X^2) - E(X)^2 = \frac{36}{35} - 1 = \frac{1}{35}.$$

4 (a) Let $X_1, .., X_5$ be the lifetimes of the 5 components.

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$N = \sum_{i=1}^{5} 1(X_i \ge 4)$ (writing $1(A)$ for 1_A)

$X_1 .. X_5$ independent $\Rightarrow$ the Bernoulli rvs $1(X_i \ge 4)$ are independent

For example $P(1(X_1\ge 4)=1,\ 1(X_2\ge 4)=0,\ 1(X_3\ge 4)=1,\ 1(X_4\ge 4)=0,\ 1(X_5\ge 4)=1)$

$= P(X_1\ge 4,\ X_2<4,\ X_3\ge 4;\ X_4<4,\ X_5\ge 3)$

$= P(X_1\ge 4)\,P(X_2<4)\,P(X_3\ge 4)\,P(X_4<4)\,P(X_5\ge 3)$

$= P(1(X_1\ge 4)=1)\,P(1(X_2\ge 4)=0)\,P(1(X_3\ge 4)=1)\,P(1(X_4\ge 4)=0)$

$\times P(1(X_4\ge 4)=1)$.

So N is Binomial $n = 5$ $\quad p = P(X_i \geq 4) = \int_4^\infty \frac{1}{2} e^{-x/2} dx$

$$= \int_2^\infty e^{-y} dy = \underline{\underline{e^{-2}}}$$

$\underline{E(N) = 5e^{-2}}$

$\underline{\underline{\text{or}}}$ $\quad E(N) = E\left(\sum_1^5 1(X_i \geq 4)\right) = \sum_1^5 E(1(X_i \geq 4)) = 5P(X_i \geq 4) = 5e^{-2}$.

(b) $P(\text{system working 4 years}) = P(N \geq 2)$

$$= 1 - P(N=0) - P(N=1)$$

$$= 1 - \binom{5}{0}(e^{-2})^0(1-e^{-2})^5 - \binom{5}{1}(e^{-2})(1-e^{-2})^4$$

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(by N binomial$(5, e^{-2})$)

$$= 1 - (1-e^{-2})^4\left[1 - e^{-2} + 5e^{-2}\right]$$

$$= 1 - (1-e^{-2})^4\left[1 + 4e^{-2}\right]$$

$$\doteq \underline{.1384}$$

5. (a) (i) $F(x) = \int_{-\infty}^x f(y)\,dy = \int_0^x \frac{1}{10} dy = \frac{x}{10}$ if $x \in [0,10)$

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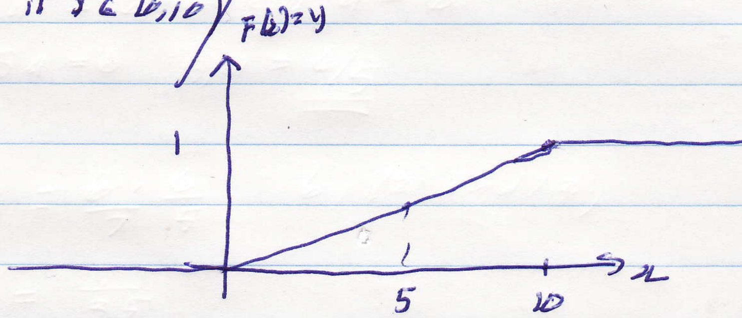
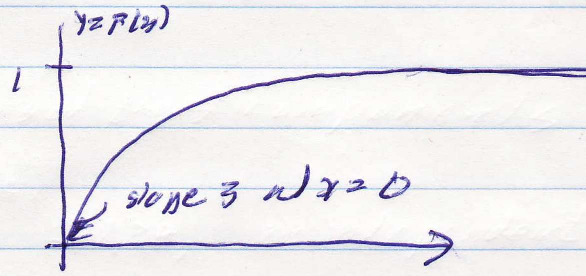
$$\therefore F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{10} & \text{if } x \in [0,10) \\ 1 & \text{if } x \geq 10 \end{cases}$$



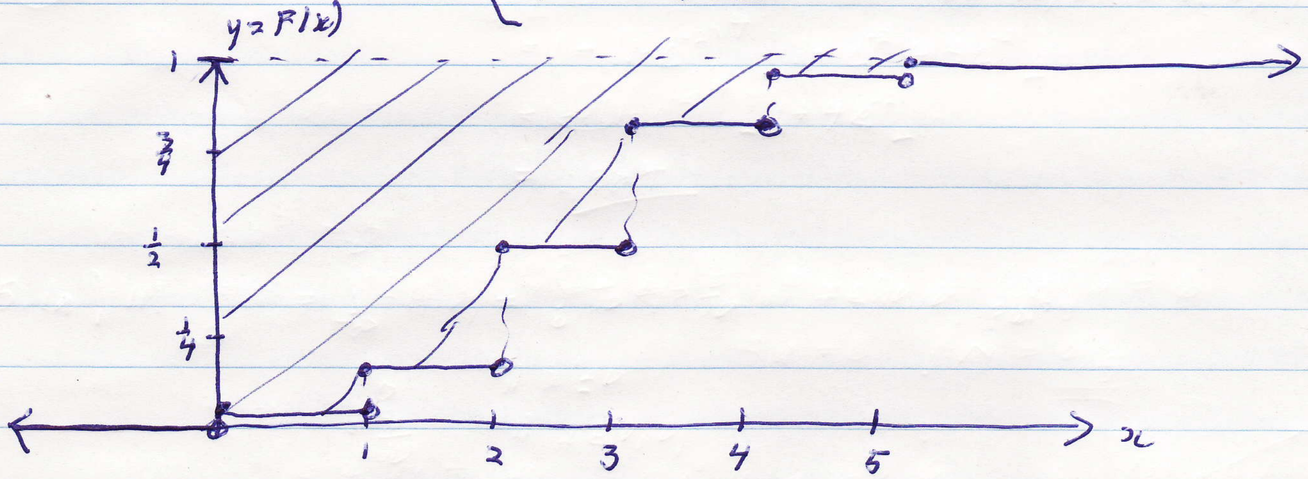
(ii) $F(x) = \int_0^x 3e^{-3y}\,dy = 1 - e^{-3x}$ if $x > 0$

$$\therefore F(x) = \begin{cases} 1 - e^{-3x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$



(iii) $P(X=k) = \binom{5}{k}\left(\frac{1}{2}\right)^5 = \binom{5}{k}\frac{1}{32} = \begin{cases} \frac{1}{32} & k=0 \text{ or } 5 \\ \frac{5}{32} & k=1 \text{ or } 4 \\ \frac{10}{32} & k=2 \text{ or } 3 \end{cases}$

$F(x) = \sum_{0 \le k \le x} P(X=k) = \begin{cases} 0 & x<0 \\ \frac{1}{32} & 0 \le x < 1 \\ \frac{6}{32} & 1 \le x < 2 \\ \frac{16}{32} & 2 \le x < 3 \\ \frac{26}{32} & 3 \le x < 4 \\ \frac{31}{32} & 4 \le x < 5 \\ 1 & 5 \le x \end{cases}$



(b) (iii) $\int_0^\infty P(X>x)\,dx = \int_0^\infty 1 - F(x)\,dx =$ area of shaded region in above

(2) $= \dfrac{31+26+16+6+1}{32} = \dfrac{80}{32} = \dfrac{5}{2} = np = E(X)$

(1) (ii) $\int_0^\infty P(X>x)\,dx = \int_0^\infty e^{-3x}\,dx = \frac{1}{3}\int_0^\infty e^{-3x}3\,dx = \frac{1}{3} = E(X)$

(1) (i) $\int_0^\infty P(X>x)\,dx = \int_0^{10} 1 - \frac{x}{10}\,dx = 10 - \frac{1}{10}\cdot\frac{10^2}{2} = 10-5=5 = \frac{10+0}{2} = E(X).$

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