MATH 302 SOLUTIONS TO HW 4

1. (i) X = no. of sales is binomial $n=100$ $p=.03$

$P(X \le 1) = \binom{100}{0}(.97)^{100} + \binom{100}{1}(.03)(.97)^{99}$

$= .97^{100} + 3(.97)^{99} \approx .1946$

(2) So P(at least 2 sales) = $1 - P(X \le 1) \approx \underline{.8054}$

(ii) X is approximately Poisson $\lambda = np = 3$.

(2) $P(X \le 1) \approx \frac{e^{-3}}{0!} + \frac{e^{-3} \cdot 3}{1!} = 4 \cdot e^{-3} = .1991$

P(at least 2 sales) $\approx 1 - P(X \le 1) = \underline{.8008}$.

2. We seek n so that if X is binomial $(n, .01)$
$P(X \ge 1) \ge .95$ or equivalently
$P(X = 0) \le .05$.

(3) If n is s.t. $n \times .01$ is moderate, the Poisson Paradigm would apply and so we want

$P(X=0) \approx e^{-n(.01)} \le .05 \iff e^{n/100} \ge 20$

$\iff n/100 \ge \log(20)$

$\iff n \ge 100(\log(20)) = 299.57$

So $\underline{\underline{n \ge 300}}$ will do. Note that $\lambda = n \cdot p = \frac{300}{100} = 3$

is moderate.

3. (a) $P(X > m+n \mid X > m) = \frac{P(X > m+n)}{P(X > m)} = \frac{(1-p)^{m+n}}{(1-p)^m}$

(2) $= (1-p)^n = P(X > n)$.

(b) $E(r^X) = \sum_{n=1}^{\infty} r^n (1-p)^{n-1} p = rp \sum_{n=1}^{\infty} (r(1-p))^{n-1}$

(2)

(by (GSO)) $\longrightarrow$ $= \frac{rp}{1 - r(1-p)}$ for $0 < r < \frac{1}{1-p}$

(c) Apply (b) with r^k in place of r to see

$$k^{th} \text{ moment of } r^X = E((r^X)^k) = E((r^k)^X)$$

$$= \frac{r^k p}{1-r^k(1-p)} \quad \text{for} \quad 0 \le r^k < \frac{1}{1-p}.$$

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ie k^{th} moment $= \dfrac{r^k p}{1-r^k+pr^k}$ for $0 \le r < (1-p)^{-\frac{1}{k}}$.

$\therefore \operatorname{Var}(r^X) = E((r^X)^2) - E(r^X)^2 = \dfrac{r^2 p}{1-r^2+pr^2} - \dfrac{r^2p^2}{[1-r(1-p)]^2}$

4. (a) $E((X-c)^2) = E(X^2-2cX+c^2) = E(X^2) - 2cE(X) + c^2$

$= E(X^2) + c^2 \quad \because \quad E(X) = 0$

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$> E(X^2) \quad$ for $c \neq 0$

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(b) Let $\hat{X} = X-\mu$. Then $E(\hat{X}) = E(X) - \mu = 0$.

$E((X-c)^2) = E(\hat{X}-(c-\mu))^2 > E(\hat{X}^2)$ for all $c \neq \mu$ by (a) applied to $\hat{X}$

$\underline{= E((X-\mu)^2)}$

5. $E(X+Y) = E(X) + E(Y) = \frac{1}{2} + \frac{1}{2} = 1$.

$P(|X-Y| = 0) = P(X=Y=1) + P(X=Y=-1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$P(|X-Y|=1) = P(X=1, Y=-1) + P(X=-1, Y=1) = \frac{1}{2}$

$\therefore E(|X-Y|) = 1 \cdot \frac{1}{2} = \frac{1}{2}$.

(1) $E(X+Y)\, E(|X-Y|) = 1 \cdot 1/2 = 1/2$

$X|X-Y| = \begin{cases} 1 & \text{if } X=1,\ Y=0 \\ 0 & \text{otherwise} \end{cases}$

$\therefore E(X|X-Y|) = 1 \cdot P(X=1, Y=0) = 1/4$. Similarly $E(Y|X-Y|) = 1/4$

So (2) $E((X+Y)|X-Y|) = E(X|X-Y|) + E(Y|X-Y|) = 1/4 + 1/4 = 1/2$.

~~E((X+Y)|X-Y~~

Comparing (1) and (2) gives $E((X+Y)|X-Y|) = E((X+Y))\, E(|X-Y|) = \frac{1}{2}$

$\therefore$ X+Y and |X−Y| are uncorrelated.

$P(X+Y=2,\ |X-Y|=1) = P(X=1, Y=1,\ |X-Y|=1) = 0.$

$P(X+Y=2)\,P(|X-Y|=1) = P(X=1,Y=1)\left[P(X=1,Y=0) + P(X=0,Y=1)\right]$

5 $= \frac{1}{4}\left[\frac{1}{2}\right] = 1/8.$

So $P(X+Y=2,\ |X-Y|=1) \neq P(X+Y=2)\,P(|X-Y|=1)$

$\therefore X+Y$ and $|X-Y|$ are dependent.

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