MATH 302 Assignment 3 Solutions

Points

1. $B_i = \{\text{select coin } i\}$ $i=1,2$; $A = \{\text{get exactly 3 H's}\}$

(a) $P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2)$

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$$= \left[\binom{5}{3}\left(\tfrac{1}{3}\right)^3\left(\tfrac{2}{3}\right)^2\tfrac{1}{2} + \binom{5}{3}\left(\tfrac{3}{4}\right)^3\left(\tfrac{1}{4}\right)^2\tfrac{1}{2}\right]$$

$$= \frac{20}{243} + \frac{135}{1024} \approx .1828$$

(b) $P(B_2|A) = \dfrac{P(A|B_2)P(B_2)}{P(A)} \approx \dfrac{135/1024}{.1828} \approx .7213$

2. <u>See End</u>

3. $P(Z=1) = P(X=Y=1) + P(X=Y=-1) = \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{2}$

$P(Z=-1) = 1 - P(Z=1) = \tfrac{1}{2}$.

X, Y, Z each take on the 2 values ± 1.

So to check X, Z are independent it suffices to check

$P(X=x, Z=z) = P(X=x)P(Z=z)$ for $x=\pm1, z=\pm1$ ie. 4 values

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The RHS is always $\tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{4}$.

$P(X=1, Z=1) = P(X=1, XY=1) = P(X=1, Y=1) = P(X=1)P(Y=1) = \tfrac{1}{4}$

$P(X=1, Z=-1) = P(X=1, XY=-1) = P(X=1, Y=-1) = \tfrac{1}{4}$

$P(X=-1, Z=1) = P(X=-1, XY=1) = P(X=-1, Y=-1) = 1/4$

$P(X=-1, Z=-1) = P(X=-1, Y=1) = 1/4$.

$\therefore X, Z$ are independent

By symmetry Y and Z are independent

By hypothesis X and Y are independent.

$\therefore X, Y, Z$ are pairwise ind't.

But $P(X=1, Y=1, Z=1) = P(X=1, Y=1) = P(X=1)P(Y=1) = \tfrac{1}{4}$

$P(X=1)P(Y=1)P(Z=1) = \left(\tfrac{1}{2}\right)^3 = 1/8$

So $P(X=1, Y=1, Z=1) \neq P(X=1)P(Y=1)P(Z=1)$

$\therefore X, Y, Z$ are <u>not</u> independent

4 (a) $P(A_1 \cup A_2 \cup A_3) = 1 - P((A_1 \cup A_2 \cup A_3)^c)$

$= 1 - P(A_1^c \cap A_2^c \cap A_3^c)$ (De Morgan's Law)

$= 1 - P(A_1^c)P(A_2^c)P(A_3^c)$ (independence)

2 $= 1 - \frac{9}{10} \times \frac{49}{50} \times \frac{99}{100}$

$= 1 - \frac{43,659}{50,000} = \underline{.12682}$

(b) By Inclusion - Exclusion

$P(A_1 \cup A_2 \cup A_3) = \sum_{i=1}^{3} P(A_i) - \sum_{1 \le i < j \le 3} P(A_i \cap A_j) + P(A_1 \cap A_2 \cap A_3)$

$= .13 - P(A_1)P(A_2) - P(A_1)P(A_3) - P(A_2)P(A_3)$

4 $+ P(A_1 \cap A_2 \cap A_3)$

(by pairwise independence)

(1) $= .13 - \frac{1}{500} - \frac{1}{1000} - \frac{1}{5000} + P(A_1 \cap A_2 \cap A_3)$

Now (2) $0 \le P(A_1 \cap A_2 \cap A_3) \le P(A_2 \cap A_3) = P(A_2)P(A_3) = \frac{1}{5000}$

Use the lower bound in (2) to see that (1) implies

$P(A_1 \cup A_2 \cup A_3) \ge .13 - \frac{1}{500} - \frac{1}{1000} - \frac{1}{5000} = \frac{634}{5000} = \underline{\underline{.1268}}$

Use the upper bound in (2) to see that (1) implies

$P(A_1 \cup A_2 \cup A_3) \le .13 - \frac{1}{500} - \frac{1}{1000} - \frac{1}{5000} + \frac{1}{5000} = .13 - .003 = \underline{\underline{.127}}$

5. By the binomial theorem

$$0=(-1+1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k (1)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k .$$

6. Let $X = \#$ of H's in n tosses.

$$P(X \text{ even}) - P(X \text{ odd}) = \sum_{k=0,\, k \text{ even}}^{n} \binom{n}{k}\left(\tfrac{1}{2}\right)^n - \sum_{k=0,\, k \text{ odd}}^{n} \binom{n}{k}\left(\tfrac{1}{2^n}\right)$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^{-n}$$

$$= 0 \quad \text{by no. 5}.$$

$\therefore P(X \text{ even}) = P(X \text{ odd}) = P(\{X \text{ even}\}^c) = 1 - P(X \text{ even})$

$\therefore 2P(X \text{ even}) = 1$

$\therefore P(X \text{ even}) = \frac{1}{2}$ and so $P(X \text{ odd}) = \frac{1}{2}$.

2. Let $X = \#$ of incorrectly transmitted bits among the 5 sent.
X is Binomial $(n=5, p=.2)$.

$$P(\text{correct transmission}) = P(X \le 2)$$

$$= \sum_{k=0}^{2} \binom{5}{k} (.2)^k (.8)^{5-k}$$

$$= .94208$$

So the protocol has increased the probability of a correct transmission from .8 to .94208.