Solutions to Assignment 10

1. (a) X = # supporting party A in poll is Binomial $n=1350$, $p=.4$

$$P(X>544)=P(X\geq 544.5)$$
$$=P\left(\frac{X-(1350)(\frac{2}{5})}{\sqrt{1350}\sqrt{\frac{2}{5}\frac{3}{5}}}\geq\frac{544.5-(1350)(2/5)}{\sqrt{1350}\sqrt{\frac{2}{5}\frac{3}{5}}}\right)$$
$$\approx P(Z\geq .25)=1-\Phi(.25)=1-.5987=.4013$$

(b) Y = # supporting party B. Y is Binomial $n=1350$ $p=.002$

$np=2.7$. Y is approximately Poisson $\lambda=2.7$

$$P(Y\leq 2)\approx e^{-2.7}+2.7e^{-2.7}+\frac{(2.7)^2}{2}e^{-2.7}$$
$$=e^{-2.7}\left[3.7+\frac{(2.7)^2}{2}\right]\doteq .49$$

2. (a) $M_{X_n}(t)=(e^tp_n+(1-p_n))^n=[1+p_n(e^t-1))^n$.

(b) Recall $(*)$ $\lim x_n n=x \Rightarrow \lim(1+x_n)^n=e^x$

Apply this with $x_n=p_n(e^t-1)$.

$$\lim x_n n=\lim np_n(e^t-1)=\lambda(e^t-1)$$
$$\therefore \lim M_{X_n}(t)=\lim_{n\to\infty}[1+x_n]^n=e^{\lambda(e^t-1)}=M_X(t),\ X \text{ Poisson } (\lambda)$$

3. (a) $$M_{X_1}(t)=E(e^{tX_1})=\sum_{n=1}^{\infty}e^{tn}p(1-p)^{n-1}=pe^t\sum_{n=1}^{\infty}[e^t(1-p)]^{n-1}$$
$$=\frac{pe^t}{1-e^t(1-p)}\quad \text{for } e^t<\frac{1}{1-p}\iff t<\ln\left(\frac{1}{1-p}\right)$$

$$M_{\bar{X}_n}(t)=E\left(e^{\frac{t}{n}(X_1+\dots+X_n)}\right)=M_{X_1+\dots+X_n}(t/n)=M_{X_1}(t/n)^n$$
$$=\left[\frac{pe^{t/n}}{1-e^{t/n}(1-p)}\right]^n,\ \frac{t}{n}<\ln\left(\frac{1}{1-p}\right)\iff t<n\ln\left(\frac{1}{1-p}\right).$$

(b) By the Weak Law of Large No.'s $\bar{X}_n$ will be close to $\frac{1}{p}$ with high probability so we expect $\lim_{n\to\infty}E(e^{t\bar{X}_n})=e^{t/p}$.

(c) Set $1+x_n = \dfrac{pe^{t/n}}{1-e^{t/n}(1-p)}$, so $x_n = \dfrac{pe^{t/n}-1+e^{t/n}-pe^{t/n}}{1-e^{t/n}+pe^{t/n}}$

$$= \frac{-1+e^{t/n}}{1-e^{t/n}+pe^{t/n}},$$

$\therefore n x_n = \dfrac{n[e^{t/n}-1]}{1-e^{t/n}+pe^{t/n}} \xrightarrow[n\to\infty]{} \dfrac{t}{p}$ $\quad \because n(e^{t/n}-1) = \left(\dfrac{e^{t/n}-1}{t/n}\right)t \to t$

So by (x) in 2(b),

$$\lim_{n\to\infty} M_{\bar{X}_n}(t) = \lim_{n\to\infty}(1+x_n)^n = e^{t/p}, \text{ as expected.}$$

4. $E(e^{X_1+X_2+X_3}) = E(e^{X_1})\,E(e^{X_2})\,E(e^{X_3})$

$$= M_{X_1}(1)\,M_{X_2}(1)\,M_{X_3}(1)$$

$$= e^{1/2}\left(\frac{2}{2-1}\right)\left(\frac{3}{3-1}\right) = \underline{3e^{1/2}}.$$

5. $A_j = \{\text{couple } j \text{ at same table}\}$.

$P(A_j) = \sum_{i=1}^{4} P(\text{Husband at table } i \mid \text{wife at table } i)\, P(\text{wife at table } i)$

$$= \sum_{i=1}^{4} \frac{5}{23}\cdot\frac{1}{4} = \frac{5}{23}$$

$X = \#$ couples seated at same table $= \sum_{j=1}^{12} 1_{A_j}$.

$E(X) = \sum_{j=1}^{12} P(A_j) = \dfrac{12\times 5}{23} = \dfrac{60}{23}$. □