Math 5016 Assignment 1– Due Thurs. Sept. 30

1. Let \((Z^{(n)}_k : k \geq 0)\) be a critical Galton-Watson branching process with offspring variance \(\gamma \in (0, \infty)\) and initial value \(z^n_0\) satisfying \(\lim_n z^n_0/n = x_0 > 0\). Set \(Y_n = Z^{(n)}_n/n\). Let \(N\) be Poisson with mean \(2x_0/\gamma\) and \((U_i, i \geq 1)\) be iid exponential rv’s with mean \(\gamma/2\) which are independent of \(N\). Prove that \(Y_n\) converges weakly to \(\sum_{i=1}^N U_i\) by showing that for any \(\lambda \geq 0\), \(\lim_{n \to \infty} E(e^{-\lambda Y_n}) = E(e^{-\lambda \sum_{i=1}^N U_i})\). It is well-known that this implies the required weak convergence and you may assume this.

2. Complete the proof of Kolmogorov’s theorem (Thm. 1.3) and show if \(\{Z_n\}\) is a Galton-Watson process starting at \(z_0 \in \mathbb{N}\), with offspring distribution having mean 1 and finite variance \(\gamma > 0\), then
\[
P(Z_n > 0) \sim \frac{2z_0}{\gamma n}.
\]
Recall we proved this in class for \(z_0 = 1\) and you may of course use this result.

3. Assume \(f : \mathbb{R}_+ \to \mathbb{R}\) is continuous and the right-hand derivative \(\frac{d^+ f}{ds}(s)\) is continuous on \([0, \infty)\). Prove that \(f\) is continuously differentiable on \([0, \infty)\).

4. Assume \(M\) is a continuous \((\mathcal{F}_t)\)-local martingale such that \(M^*_T\) is integrable for all \(T \geq 0\). Prove that \(M\) is a continuous \((\mathcal{F}_t)\)-martingale.

5. A collection of probabilities \(\{P^x : x \in E\} \ (E \text{ Polish})\) is a Borel Markov processes if it satisfies the properties of a Borel strong Markov process, except that the strong Markov property (iii) is replaced by the ordinary Markov property. Give an example of a Borel Markov process with continuous paths which is not a Borel strong Markov process.