1. Assume $X$ is a standard $(\mathcal{F}_t)$-Poisson process with rate $\lambda > 0$.
(a) Prove that $t \to X_t$ is increasing and $\mathbb{Z}_+$-valued a.s.
(b) Prove that $\Delta X(t) = 0$ or $1$ for all $t > 0$ a.s.
(c) If $S_n = \inf\{t \geq 0 : X_t = n\}$, show that $\tau_n = S_n - S_{n-1}$ $(n \geq 1)$ is a sequence of iid exponential rv’s with mean $\lambda^{-1}$. Note that $X_t = \sum_{n=1}^{\infty} 1(S_n \leq t)$, which would give a simpler way of constructing a Poisson process of course.
(d) Verify directly that $M_t = X_t - \lambda t$, $M_t^2 - X_t$, and $M_t^2 - \lambda t$ are $(\mathcal{F}_t)$-martingales.

2. Show there is a constant $c_4$ so that for any martingale $\{M_n : n \in \mathbb{Z}_+\}$, if $d_n = M_n - M_{n-1}$ and $[M] = \sum_{n=1}^{\infty} d_n^2$ then $E([M]^2) \leq c_4 E((M^*)^4)$.

3. Let $M \in \mathcal{M}_{0,\text{loc}}$.
   (a) If $M^*_t \in L^1$ for all $t > 0$, then prove $M$ is a martingale.
   (b) Assume $M$ is continuous (but not the hypothesis in (a)).
      (i) Prove that if $E([M]_t) < \infty$ for all $t > 0$, then $M$ is a square integrable $(\mathcal{F}_t)$-martingale and $M_t^2 - [M]_t$ is an $(\mathcal{F}_t)$-martingale .
      (ii) Prove that for any $a, b > 0$, $P(M_t^* \geq a, [M]_t \leq b) \leq \frac{b}{a^2}$. Hint: One approach is to consider $M_T$, where $T = \inf\{t \geq 0 : [M]_t \geq b\}$.