S1. Proof of Proposition 5.1

For $|x_i| \geq \varepsilon_0, i = 1, 2$, and $\varepsilon \in (0, \varepsilon_0)$, if $|x_1 - x_2| \leq 5\varepsilon$, then use $x e^{-x} \leq e^{-1}, \forall x \geq 0$ to get

$$E_{\delta_0} \left( \prod_{i=1}^{2} \lambda \frac{X_{G_{\varepsilon}^{-\frac{i}{2}}}(1)}{\varepsilon^2} \exp \left( - \lambda \frac{X_{G_{\varepsilon}^{-\frac{i}{2}}}(1)}{\varepsilon^2} \right) \right) \leq e^{-1} E_{\delta_0} \left( \lambda \frac{X_{G_{\varepsilon}^{-\frac{1}{2}}}(1)}{\varepsilon^2} \exp \left( - \lambda \frac{X_{G_{\varepsilon}^{-\frac{1}{2}}}(1)}{\varepsilon^2} \right) \right).$$

Recall the definition of $F = F_{\varepsilon,x_1}$ in (4.17). For all $\lambda > 0$, an integration by parts gives

$$E_{\delta_0} \left( \lambda \frac{X_{G_{\varepsilon}^{-\frac{1}{2}}}(1)}{\varepsilon^2} \exp \left( - \lambda \frac{X_{G_{\varepsilon}^{-\frac{1}{2}}}(1)}{\varepsilon^2} \right) \right) = \int_{0}^{\infty} \lambda x e^{-\lambda x} dF(x)$$

$$= \int_{0}^{\infty} \lambda (\lambda x - 1)e^{-\lambda x} F(x) dx = \int_{0}^{\infty} (y-1) e^{-y} F \left( \frac{y}{\lambda} \right) dy \leq F(2) + \int_{2\lambda}^{\infty} ye^{-y} F \left( \frac{y}{\lambda} \right) dy$$

$$\leq c_4 g \frac{2^{p-2}}{\varepsilon^{p-2}} + \int_{2\lambda}^{\infty} ye^{-y} c_4 g \frac{(\frac{y}{\lambda})^{p-2}}{\varepsilon^{p-2}} dy = C(\varepsilon_0, \lambda) \varepsilon^{p-2},$$

the last line by Proposition 4.9. Therefore

$$E_{\delta_0} \left( \prod_{i=1}^{2} \lambda \frac{X_{G_{\varepsilon}^{-\frac{i}{2}}}(1)}{\varepsilon^2} \exp \left( - \lambda \frac{X_{G_{\varepsilon}^{-\frac{i}{2}}}(1)}{\varepsilon^2} \right) \right) \leq e^{-1} C(\varepsilon_0, \lambda) \varepsilon^{p-2}$$

$$\leq e^{-1} 5^{p-2} C(\varepsilon_0, \lambda) \varepsilon^{p-2} = e^{-1} 5^{p-2} C(\varepsilon_0, \lambda) |x_1 - x_2|^{2-p} \varepsilon^{2(p-2)},$$

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provided \(|x_1 - x_2| \leq 5\varepsilon\). As a result,

throughout the rest of this Section we may fix \(\varepsilon_0 > 0\), \(|x_i| \geq \varepsilon_0\) and \(\varepsilon \in (0, \varepsilon_0)\) with \(|x_1 - x_2| > 5\varepsilon\). In this case, we have \(B(x_1, 2\varepsilon) \cap B(x_2, 2\varepsilon) = \emptyset\).

Let \(\vec{x} = (x_1, x_2)\), \(G = G_{\varepsilon_i}^1 \cap G_{\varepsilon_i}^2\), and \(\vec{\lambda} = (\lambda_1, \lambda_2) \in [0, \infty)^2 \setminus \{(0, 0)\}\). For \(X_0 \in MF(\mathbb{R}^d)\) such that \(d(\text{Supp}(X_0), G) > 0\), the decomposition (2.4) with \(G = G_{\varepsilon_i}^i\), \(i = 1, 2\), gives

\[\mathbb{E}_x \left( \exp \left( - \sum_{i=1}^{2} \lambda_i \frac{X_{\varepsilon_i}^G(1)}{\varepsilon^2} \right) \right) = \exp \left( - \int U^\vec{\lambda},\vec{x},\varepsilon(x) X_0(dx) \right),\]

where \(U^\vec{\lambda},\vec{x},\varepsilon \geq 0\) is defined as

\[U^\vec{\lambda},\vec{x},\varepsilon(x) \equiv N_x(1 - \exp \left( - \sum_{i=1}^{2} \lambda_i \frac{X_{\varepsilon_i}^G(1)}{\varepsilon^2} \right)), \quad \forall x \in G.\]

We use results from Chapter V of [17] to get the following lemma.

**Lemma S.1.1.** \(U^\vec{\lambda},\vec{x},\varepsilon\) is a \(C^2\) function on \(G\) and solves

\[\Delta U^\vec{\lambda},\vec{x},\varepsilon = (U^\vec{\lambda},\vec{x},\varepsilon)^2 \text{ on } G.\]

Moreover,

\[U^\vec{\lambda},\vec{x},\varepsilon(x) \leq (\lambda_1 + \lambda_2)\varepsilon^{-2}, \quad \forall x \in G.\]

**Proof.** Let

\[u(x) \equiv U^\vec{\lambda},\vec{x},\varepsilon(x) = N_x(1 - \exp \left( - \sum_{i=1}^{2} \lambda_i \frac{X_{\varepsilon_i}^G(1)}{\varepsilon^2} \right)).\]

Then use \(1 - e^{-x} \leq x\) to get

\[u(x) \leq N_x \left( \sum_{i=1}^{2} \lambda_i \frac{X_{\varepsilon_i}^G(1)}{\varepsilon^2} \right) = \sum_{i=1}^{2} \lambda_i \varepsilon^{-2} P_x(\tau_i < \infty) \leq (\lambda_1 + \lambda_2)\varepsilon^{-2},\]

the equality by Proposition V.3 of [17], where \((B_t)\) is \(d\)-dimensional Brownian motion starting from \(x\) under \(P_x\) and \(\tau_i = \inf\{t \geq 0 : B_t \notin G_{\varepsilon_i}^i\}\).
Next, for any $x' \in G$, let $D$ be an open ball that contains $x'$, whose closure is in $G$. Use (S.1) with $X_0 = \delta_x$ and then Proposition 2.3(b)(i) to see that for $x \in D$,

$$e^{-u(x)} = \mathbb{E}_{\delta_x} \left( \exp \left( - \sum_{i=1}^{2} \lambda_i \frac{X_{G_x}^{i}(1)}{\varepsilon^2} \right) \right) = \mathbb{E}_{\delta_x} \left( \mathbb{E}_{X_D} \left( \exp \left( - \sum_{i=1}^{2} \lambda_i \frac{X_{G_x}^{i}(1)}{\varepsilon^2} \right) \right) \right)$$

$$= \mathbb{E}_{\delta_x} \left( \exp \left( - \int u(x) X_D(dx) \right) \right) = \exp \left( - \mathbb{E}_{\delta_x} \left( 1 - \exp \left( - \int u(y) X_D(dy) \right) \right) \right),$$

the third equality by (S.1) with $X_0 = X_D$, and the last by the decomposition (2.4). Therefore

$$u(x) = \mathbb{E}_{\delta_x} \left( 1 - \exp \left( - \int u(y) X_D(dy) \right) \right) \quad \forall x \in D.$$

Note $u$ is bounded in $G$ by (S.4), and hence on $\partial D$. Use Theorem V.6 of [17] to conclude

$$\Delta u(x) = (u(x))^2, \; \forall x \in D, \; \text{and, in particular, for } x = x'.$$

Since $x'$ is arbitrary, it holds for all $x \in G$. \hfill \blacksquare

Let $X_0 = \delta_x$ in (S.1) for $x \in G$ to get

(S.5) $$\mathbb{E}_{\delta_x} \left( \exp \left( - \sum_{i=1}^{2} \lambda_i \frac{X_{G_x}^{i}(1)}{\varepsilon^2} \right) \right) = \exp(-U_{\lambda, x, \varepsilon}(x)).$$

Monotone convergence and the convexity of $e^{-ax}$ for $a, x > 0$ allow us to differentiate the left-hand side of (S.5) with respect to $\lambda_i > 0$ through the expectation and so conclude that for $i = 1, 2$, $U_{i, \lambda, x, \varepsilon}(x) = \frac{\partial}{\partial \lambda_i} U_{\lambda, x, \varepsilon}(x)$ exists and

$$\mathbb{E}_{\delta_x} \left( \frac{X_{G_x}^{i}(1)}{\varepsilon^2} \exp \left( - \sum_{i=1}^{2} \lambda_i \frac{X_{G_x}^{i}(1)}{\varepsilon^2} \right) \right) = e^{-U_{i, \lambda, x, \varepsilon}(x)} U_{i, \lambda, x, \varepsilon}(x) \text{ for } \lambda_1 > 0, \lambda_2 > 0.$$

Repeat the above to see that $U_{\lambda, x, \varepsilon}(x)$ is $C^2$ in $\lambda_1, \lambda_2 > 0$ and if

$$U_{1, 2, \lambda, x, \varepsilon}(x) = \frac{\partial^2}{\partial \lambda_1 \partial \lambda_2} U_{\lambda, x, \varepsilon}(x),$$

then

(S.6) $$\mathbb{E}_{\delta_x} \left( \frac{X_{G_x}^{1}(1)}{\varepsilon^2} \frac{X_{G_x}^{2}(1)}{\varepsilon^2} \exp \left( - \sum_{i=1}^{2} \lambda_i \frac{X_{G_x}^{i}(1)}{\varepsilon^2} \right) \right)$$

$$= e^{-U_{\lambda, x, \varepsilon}(x)} \left[ U_{1, \lambda, x, \varepsilon}(x) U_{2, \lambda, x, \varepsilon}(x) - U_{1, 2, \lambda, x, \varepsilon}(x) \right], \text{ for } \lambda_1, \lambda_2 > 0.$$

The next monotonicity result follows just as in the proof of Lemma 9.2 of [20].
**Lemma S.1.2.**

(a) $U_{i}^{\tilde{\lambda},\tilde{x},\varepsilon}(x) > 0$ is strictly decreasing in $\tilde{\lambda} \in \{ (\lambda_1, \lambda_2) : \lambda_i > 0, \lambda_{3-i} \geq 0 \}$, for $i = 1, 2$.

(b) $-U_{1,2}^{\tilde{\lambda},\tilde{x},\varepsilon}(x) > 0$ is strictly decreasing in $\tilde{\lambda} \in (0, \infty)^2$.

Note that

\[(S.7) \quad U^{\tilde{\lambda},\tilde{x},\varepsilon}(x) = U^{\lambda_i \varepsilon,-2,\varepsilon}(x - x_i), \text{ for } \lambda_i > 0 \text{ and } \lambda_{3-i} = 0.\]

The above monotonicity results easily give the following, just as for Lemma 9.3 of [20].

**Lemma S.1.3.**

(a) For all $\lambda_i > 12$ and $\lambda_{3-i} \geq 0$,

\[
U_{i}^{\tilde{\lambda},\tilde{x},\varepsilon}(x) \leq \frac{2}{\lambda_i} (U^{\lambda_i \varepsilon,-2,\varepsilon}(x_i - x) - U^{(\lambda_i/2)\varepsilon,-2,\varepsilon}(x_i - x)) \leq \frac{2}{\lambda_i} \frac{2^p}{|x_i - x|^p} D^{\lambda_i/2}(2) \varepsilon^{p-2}, \quad \forall |x_i - x| \geq 2\varepsilon.
\]

(b) For all $\lambda_1, \lambda_2 > 12$,

\[
-U_{1,2}^{\tilde{\lambda},\tilde{x},\varepsilon}(x) \leq \frac{4}{\lambda_1 \lambda_2} \min_{i=1,2} (U^{\lambda_i \varepsilon,-2,\varepsilon}(x_i - x) - U^{(\lambda_i/2)\varepsilon,-2,\varepsilon}(x_i - x)) \leq \frac{4}{\lambda_1 \lambda_2} 2^p (\lceil D^{\lambda_1/2}(2)|x_1 - x|^{-p} \rceil \land \lceil D^{\lambda_2/2}(2)|x_2 - x|^{-p} \rceil) \varepsilon^{p-2}, \quad \forall |x_i - x| \geq 2\varepsilon, \quad i = 1, 2.
\]

Let $r_\varepsilon = 2\varepsilon$ and assume $0 < r_\varepsilon < \min\{|x_i - x| : i = 1, 2\}$. Set $T_{r_\varepsilon}^i = \inf\{t \geq 0 : |B_t - x_i| \leq r_\varepsilon \}$ and $T_{r_\varepsilon} = T_{r_\varepsilon}^1 \land T_{r_\varepsilon}^2$, and let $\mathcal{F}_t$ denote the right-continuous filtration generated by the Brownian motion $B$, which starts at $x$ under $P_x$.

**Lemma S.1.4.** Let $\lambda_1, \lambda_2 > 12$.

(a) $U_{1}^{\tilde{\lambda},\tilde{x},\varepsilon}(B(t \land T_{r_\varepsilon})) - \int_0^{t \land T_{r_\varepsilon}} U_{1}^{\tilde{\lambda},\tilde{x},\varepsilon}(B(s)) U_{1}^{\tilde{\lambda},\tilde{x},\varepsilon}(B(s)) ds$ is an $\mathcal{F}_t$-martingale.

(b) For any $t > 0$,

\[
U_{1}^{\tilde{\lambda},\tilde{x},\varepsilon}(x) = E_x \left( U_{1}^{\tilde{\lambda},\tilde{x},\varepsilon}(B(t \land T_{r_\varepsilon})) \exp \left( -\int_0^{t \land T_{r_\varepsilon}} U_{1}^{\tilde{\lambda},\tilde{x},\varepsilon}(B(s)) ds \right) \right).
\]

This result follows from Lemmas S.1.1, S.1.3 and Itô’s Lemma, exactly as for Lemma 9.4 in [20], and so the proof is omitted.
LEMMA S.1.5. For all \( \lambda_1, \lambda_2 > 12 \),
\[
-U_{1,2}^\lambda(x) = E_x \left( \int_0^{T_{r_\varepsilon}} \prod_{i=1}^2 U_i^\lambda (B(t)) \exp \left( - \int_0^t U_i^\lambda (B(s)) ds \right) dt \right)
\]
\[
+ E_x \left( \exp \left( - \int_0^{T_{r_\varepsilon}} U_1^\lambda (B(s)) ds \right) 1(T_{r_\varepsilon} < \infty)(-U_{1,2}^\lambda (B(T_{r_\varepsilon}))) \right).
\]

This follows from Lemmas S.1.3 and S.1.4, as in the proof of Lemma 9.5 of [20].

PROOF OF PROPOSITION 5.1. Recall \( r_\varepsilon = 2\varepsilon \). For the case \( \varepsilon \in [\varepsilon_0/2, \varepsilon_0) \), the result follows immediately by letting \( c_{5.1} \geq e^{-2/\varepsilon_0^2}e^{-2(p-2)/\varepsilon_0^2} \) and by using \( xe^{-x} \leq e^{-1} \) for \( x \geq 0 \), so we assume
\[
(S.8) \quad r_\varepsilon = 2\varepsilon < \varepsilon_0.
\]
Recall that \( T_{r_\varepsilon} = \inf\{t \geq 0 : |B_t - x| \leq r_\varepsilon \} \) and \( T_{r_\varepsilon} = T_{r_\varepsilon}^1 \wedge T_{r_\varepsilon}^2 \). Since \( |x_1| \geq \varepsilon_0 \), we have \( T_{r_\varepsilon} > 0, P_{\varepsilon_0}\text{-a.s.} \). We set \( \lambda = (\lambda, \lambda) \), \( \bar{x} = (x_1, x_2) \), and \( \Delta = |x_1 - x_2| \), where the constant \( \lambda > 0 \) will be chosen large below.

Apply (S.6) and Lemma S.1.3(a) to see that for \( \lambda > 12 \),
\[
E_{\delta_0} \left( \lambda^2 \frac{X_{G_{1\varepsilon}}(1)}{e^2} \frac{X_{G_{2\varepsilon}}(1)}{e^2} \exp \left( - \lambda \sum_{i=1}^2 \frac{X_{G_{1i\varepsilon}}(1)}{e^2} \right) \right)
\]
\[
= \lambda^2 e^{-U_{1,2}^\lambda(x)} \left[ U_{1,2}^\lambda (0) - U_{1,2}^\lambda (0) \right]
\]
\[
\leq 2^{p+2}(D^{\lambda/2}(2))^2|x_1|^{-p}|x_2|^{-p} \varepsilon^{2(p-2)} - \lambda^2 U_{1,2}^\lambda (0)
\]
\[
(S.9) \quad \leq \varepsilon^2 2^{p+2}(D^{\lambda/2}(2))^2|x_1|^{-p}|x_2|^{-p} \varepsilon^{2(p-2)} + \lambda^2 (-U_{1,2}^\lambda (0)).
\]

To bound the last term, use Lemma S.1.5 to get
\[
(S.10) \quad \lambda^2 (-U_{1,2}^\lambda (0))
\]
\[
= \lambda^2 E_0 \left( \int_0^{T_{r_\varepsilon}} \prod_{i=1}^2 U_i^\lambda (B(t)) \exp \left( - \int_0^t U_i^\lambda (B(s)) ds \right) dt \right)
\]
\[
+ \lambda^2 E_0 \left( \exp \left( - \int_0^{T_{r_\varepsilon}} U_1^\lambda (B(s)) ds \right) 1(T_{r_\varepsilon} < \infty)(-U_{1,2}^\lambda (B(T_{r_\varepsilon}))) \right)
\]
\[
\equiv K_1 + K_2.
\]

We first consider \( K_2 \). On \( \{T_{r_\varepsilon} < \infty\} \) we may set \( x_\varepsilon(\omega) = B(T_{r_\varepsilon}) \) and choose \( i(\omega) \) so that \( |x_i - x_\varepsilon| \geq \Delta/2 \). By the definition of \( T_{r_\varepsilon} \), \( |x_i - x_\varepsilon| \geq r_\varepsilon = 2\varepsilon \),
and so $|x_i - x_\varepsilon| \geq \frac{1}{2}(\Delta \vee r_\varepsilon)$. Lemma S.1.3(b) and the above imply
\[
\lambda^2(-U_{1,2}^{\lambda,\varepsilon}(B(T_{r_\varepsilon}))) \leq 4 \cdot 2^p(D^{\lambda/2}(\Delta \vee r_\varepsilon)^{-p}2^p)\varepsilon^{p-2} \leq c(\Delta \vee r_\varepsilon)^{-p}\varepsilon^{p-2}.
\]
This shows that
\[
(S.11)
\]
\[
K_2 \leq c(\Delta \vee r_\varepsilon)^{-p}\varepsilon^{p-2} \sum_{i=1}^2 E_0\left(1(T_{r_\varepsilon}^i < \infty) \exp\left(-\int_{0}^{T_{r_\varepsilon}^i} U_{\lambda,\varepsilon}(B(s))ds\right)\right).
\]
Use (S.7) and Corollary 4.7(a) with $|B(s) - x_i| \geq r_\varepsilon = 2\varepsilon$ and $R = 2$ to see that
\[
U_{\lambda,\varepsilon}(B(s)) \geq U_{\lambda,\varepsilon-2}(B(s) - x_i) \geq U_{\infty,\varepsilon}(B(s) - x_i) - 2^p|B(s) - x_i|^{-p}D^{\lambda}(2)\varepsilon^{p-2}
\]
\[
(S.12)
\]
where the last follows by using (4.1) and scaling to see that $U_{\infty,\varepsilon}(x) = \varepsilon^{-2}U_{\infty,1}(x/\varepsilon) \geq \varepsilon^{-2}V^{\infty}(x/\varepsilon) = V^{\infty}(x)$ for all $|x|/\varepsilon > 1$. Let $\tau_{r_\varepsilon} = \inf\{t : |B_t| \leq r_\varepsilon\}$ and let $\mu, \nu$ be as in (4.9). Use the above in (S.11) and then use Brownian scaling to see that for $i = 1, 2$,
\[
(S.13)
\]
\[
E_0\left(1(T_{r_\varepsilon}^i < \infty) \exp\left(-\int_{0}^{T_{r_\varepsilon}^i} U_{\lambda,\varepsilon}(B(s))ds\right)\right) \\
\leq E_{-x_i}\left(1(\tau_{r_\varepsilon} < \infty) \exp\left(\int_{0}^{\tau_{r_\varepsilon}} 2^p|B(s)|^{p-2}\varepsilon^{p-2}ds\right) \exp\left(-\int_{0}^{\tau_{r_\varepsilon}} \frac{2(4-d)}{|B(s)|^{p}}ds\right)\right) \\
\leq E_{-x_i/r_\varepsilon}\left(1(\tau_1 < \infty) \exp\left(\int_{0}^{\tau_1} \frac{2^p|B(s)|^{p-2}r_\varepsilon^{p-2}}{|B(s)|^{p}}ds\right) \exp\left(-\int_{0}^{\tau_1} \frac{2(4-d)}{|B(s)|^{p}}ds\right)\right) \\
= E_{|x_i|/r_\varepsilon}\left(\exp\left(\int_{0}^{\tau_1} \frac{4D^{\lambda/2}(2)}{|B(s)|^{p}}ds\right) \tau_1 < \infty\right)(|x_i|/r_\varepsilon)^{-p},
\]
where we have used Lemma 4.5 in the last line, and recalled that $p = \nu + \mu$. Choose $\lambda > 12$ large such that
\[
2\gamma \equiv 2 \cdot 4D^{\lambda}(2) \leq 2(4 - d) < \nu^2,
\]
and then apply Lemma 4.4 to conclude that (S.13) is bounded by
\[
c_{4.4}(p, \nu)(|x_i|/r_\varepsilon)^{-p} \leq c_{4.4}(p, \nu)\varepsilon_0^{-p}r_\varepsilon^{-p}.
\]
So (S.11) becomes
\[
(S.14)
\]
\[
K_2 \leq c(\Delta \vee r_\varepsilon)^{-p}\varepsilon^{p-2}2c_{4.4}(p, \nu)\varepsilon_0^{-p}r_\varepsilon^{-p} \leq c(\varepsilon_0)^{\Delta^{-p}\varepsilon^{p-2}r_\varepsilon^{-p-2}}
\]
\[
= 2^{-p-2}c(\varepsilon_0)^{\Delta^{-p}\varepsilon^{2(p-2)}}.
\]
In view of (S.9), (S.10) and (S.14), it remains to prove

\[ K_1 \leq C(\varepsilon_0)\Delta^{2-p}e^{2(p-2)}. \]

Apply Lemma S.1.3(a) to \( K_1 \) defined in (S.10) to get

\[
K_1 \leq \lambda^2 \frac{1}{\lambda^2} (2^{p+1}e^{p-2} D^{\lambda/2}(2))^2 \\
\times E_0 \left( \int_0^{T_{r_\varepsilon}} \prod_{i=1}^2 |B_t - x_i|^{-p} \exp \left( - \int_0^t U_{\lambda, \varepsilon, \varepsilon}(B(s))ds \right) dt \right).
\]

Let \( \Delta_i = x_{3-i} - x_i \), so that \(|\Delta_i| = \Delta\). Let \( T_{r_\varepsilon} = \inf \{ t : |B_t| \leq r_\varepsilon \) or
\(|B_t - \Delta_i| \leq r_\varepsilon \}\). Apply (S.12) to see that (S.16) becomes

\[
K_1 \leq c \varepsilon^{2(p-2)} \sum_{i=1}^2 E_{-x_i} \left( \int_0^{T_{r_\varepsilon}} |B_t|^{-p} |B_t - \Delta_i|^{-p} 1(|B_t| \leq |B_t - \Delta_i|) \right) \\
\times \exp \left( \int_0^t 2^p D^{\lambda}(2)e^{p-2} \frac{ds}{|B(s)|^p} \right) \exp \left( - \int_0^t 2(4-d) \frac{ds}{|B(s)|^2} \right) dt.
\]

On \( \{ |B_t| \leq |B_t - \Delta_i| \} \), we have

\[
\Delta = |\Delta_i| \leq |B_t - \Delta_i| + |B_t| \leq 2|B_t - \Delta_i|,
\]

and hence

\[
|B_t - \Delta_i|^{-p} \leq \left( \frac{1}{2} \Delta \vee |B_t| \right)^{-p} \leq 2^p (\Delta^{-p} \wedge |B_t|^{-p}).
\]

Use \( T_{r_\varepsilon} \leq \tau_{r_\varepsilon} \) and Brownian scaling to see that

\[
K_1 \leq c \varepsilon^{2(p-2)} \sum_{i=1}^2 E_{-x_i} \left( \int_0^{T_{r_\varepsilon}} |B_t|^{-p} (|B_t|^{-p} \wedge \Delta^{-p}) \right) \\
\times \exp \left( \int_0^t 2^p D^{\lambda}(2)e^{p-2} \frac{ds}{|B(s)|^p} \right) \exp \left( - \int_0^t 2(4-d) \frac{ds}{|B(s)|^2} \right) dt
\]

\[
\leq c \varepsilon^{2(p-2)} \sum_{i=1}^2 E_{-x_i/r_\varepsilon} \left( \int_0^{\tau_{r_\varepsilon}} r_\varepsilon^{2-2p} |B_t|^{-p} (|B_t|^{-p} \wedge (\Delta/r_\varepsilon)^{-p}) \right) \\
\times \exp \left( \int_0^t 2^p D^{\lambda}(2)e^{p-2} \frac{r_\varepsilon^{2-2p}}{|B(s)|^p} ds \right) \exp \left( - \int_0^t 2(4-d) \frac{ds}{|B(s)|^2} \right) dt
\]

\[
= c \varepsilon^{-2} \sum_{i=1}^2 \int_0^\infty E_{-x_i/r_\varepsilon} \left( 1(t < \tau_{r_\varepsilon}) |B(t \wedge \tau_{r_\varepsilon})|^{-p} (|B(t \wedge \tau_{r_\varepsilon})|^{-p} \wedge (\Delta/r_\varepsilon)^{-p}) \right) \\
\times \exp \left( \int_0^{t \wedge \tau_{r_\varepsilon}} 2^p D^{\lambda}(2) \frac{ds}{|B(s)|^p} \right) \exp \left( - \int_0^{t \wedge \tau_{r_\varepsilon}} 2(4-d) \frac{ds}{|B(s)|^2} \right) dt.
\]
Now let \( \delta = 4D^\lambda(2) \), \( \mu, \nu \) be as in (4.9), and use Lemma A.1 to get (S.19)

\[
K_1 \leq c \varepsilon^{-2} \sum_{i=1}^{2} \int_0^\infty (|x_i|/r \varepsilon)^{\nu-\mu} E_{|x_i|/r \varepsilon}^{(2+2 \nu)} \left( 1(t < \tau_1) \rho(t \wedge \tau_1)^{-\mu} \rho(t \wedge \tau_1)^{-\nu+\mu} \right) dt \]

\[
= c \varepsilon^{\mu-\nu-2} \sum_{i=1}^{2} |x_i|^{\nu-\mu} E_{|x_i|/r \varepsilon}^{(2+2 \nu)} \left( \int_0^{\tau_1} \rho_t^{-p-\nu+\mu} (\rho_t^{-p} \wedge (\Delta/r \varepsilon)^{-p}) \exp \left( \int_0^t \delta \rho_s^{-p} ds \right) \right) dt.
\]

We interrupt the proof of the proposition for another auxiliary result from [20].

**Lemma S.1.6.** There is some universal constant \( c_{S.1.6} > 0 \) such that for any \( r > 0 \) with \( r < (|x_i| \wedge \Delta) \) and \( 0 < \delta < (p - 2)(2 - \mu) \), we have

\[
E_{|x_i|/r}^{(2+2 \nu)} \left( \int_0^{\tau_1} \rho_t^{-p-\nu+\mu} (\rho_t^{-p} \wedge (\Delta/r \varepsilon)^{-p}) \exp \left( \int_0^t \delta \rho_s^{-p} ds \right) dt \right) \leq c_{S.1.6} r^{-2+2p+\nu-\mu} |x_i|^{-2 \nu} \Delta^2 - p.
\]

**Proof.** This is included in the proof of Proposition 6.1 of [20] with \( r = r_\lambda \). In particular, the above expectation appears in (9.23) of [20] and is bounded by \( eJ_i \) in (9.27) of that paper. Following the inequalities in that work, noting we only need consider Case 1 or Case 3 (the latter with \( r \leq |x_i| \leq \Delta \)) at the end of the proof, we arrive at the above bound. \( \blacksquare \)

Returning now to the proof of Proposition 5.1. Pick \( \lambda > 12 \) large such that \( \delta < (p - 2)(2 - \mu) \). Note we assumed \( |x_i| \geq \varepsilon_0 > r \varepsilon \) by (S.8) and \( \Delta = |x_1 - x_2| > 5 \varepsilon > r \varepsilon \) at the very beginning of this section. So use Lemma S.1.6 applied with \( r = r \varepsilon \) to see that

\[
K_1 \leq c \varepsilon^{-\nu-2} \sum_{i=1}^{2} |x_i|^{\nu-\mu} c_{S.1.6} r_{\varepsilon}^{-2+2p+\nu-\mu} |x_i|^{-2 \nu} \Delta^2 - p \]

(S.20)

\[
= C \varepsilon^{2p-4} \Delta^2 - p \sum_{i=1}^{2} |x_i|^{-p}.
\]

Use \( |x_i| \geq \varepsilon_0 \) to conclude

\[
K_1 \leq 2 C \varepsilon_0^{-p} \Delta^2 - p \varepsilon_0^{-2p-4}.
\]

This gives (S.15), and so the proof is complete. \( \blacksquare \)
S.2. Proof of Lemma 7.3. We work under $Q_{x_0}$ where $|x_0| \geq 2r_0$. Recall the definitions of $\eta^G_s$ and $E_G$ from Section 2. For $0 \leq r < r_0$, introduce

$$A^r_t = \int_0^t 1(\zeta_u \leq S_{Gr_0-r}(W_u)) \, du,$$

so that

$$\eta^r_s := \eta^G_{r_0-r} = \inf \{ t : A^r_t > s \}.$$

**Lemma S.2.1.**

(a) $Q_{x_0}$-a.s. for all $t \geq 0$ we have

$$A^r_t = \int_0^t 1(\inf_{v \leq \zeta_u} |W_u(v)| > r_0 - r) \, du \quad \forall r \in [0, r_0),$$

and

$$r \mapsto A^r_t$$

is left-continuous on $[0, r_0)$.

(b) $\lim_{r' \uparrow r} \eta^{r'}_s = \eta^r_s$ for all $r \in (0, r_0), s \geq 0$ $Q_{x_0}$-a.s.

(c) If $T$ is an $(E^+_r)$-stopping time, then $W_{\eta^r_T}$ is $E^+_T$-measurable.

**Proof.** The proof is a straightforward modification of that of Lemma 7.4 in [20], where shrinking half spaces have now been replaced with shrinking balls.

**Proof of Lemma 7.3.** By (7.23) (with a different radii) and Lemma 2.1(a) there are Borel maps $\tilde{\psi}$ on $K$ and $\psi$ on $C([0, \infty), W)$ such that

$$1_{D_{r_0}} = \tilde{\psi}(R) = \lim_{N \to \infty} \tilde{\psi}(\{ W(s) : s \leq N \}) = \psi(W),$$

where we have used (2.2) in the second equality. In the last equality we have also called on the continuity of $W \mapsto \{ W(s) : s \leq N \}$ from $C([0, \infty), W)$ to $K$. Therefore a monotone class argument shows it suffices to fix $s \geq 0$ and show that if $\phi : W \to \mathbb{R}$ is bounded Borel then

$$\phi(W_s)$$

is $E^+_{T_0-}$-measurable.

Lemma S.2.1(b) implies that $W_{\eta^r_{n_0}} = \lim_{n \to \infty} W_{\eta^r_{n-1}} Q_{x_0}$-as. and so by Lemma S.2.1(c) and (7.20), $W_{\eta^r_{T_0}}$ is $E^+_{T_0-}$-measurable. So to prove (S.21) it suffices to show

$$W_s = W_{\eta^r_{T_0}} Q_{x_0} - a.s.$$

This, in turn, would follow from $A^r_{T_0} = t$ for all $t \geq 0 Q_{x_0}$-a.s., or equivalently by Lemma S.2.1(a),

$$\int_0^\sigma 1(\inf_{v \leq \zeta_u} |W_u(v)| \leq r_0 - T_0) \, du = 0 \quad Q_{x_0} - a.s..$$
Here we have truncated the integral at $\sigma$ since $\zeta_u = 0$ and $|W_u(0)| = |x_0| \geq 2r_0$ for $u \geq \sigma$. If $0 \leq u < \zeta_s$ and $s' < s$ is the last time before $s$ that $\zeta_{s'} = u$, then $\inf_{t \in [s', s]} \zeta_{\bar{t}} = \zeta_{s'} = u$ and so (e.g., see p. 66 of [17]) $W_s(u) = \hat{W}(s')$ $Q_{x_0}$-a.s. This and Lemma 7.1 (recall also (7.1)) imply

$$\inf_{u \leq \sigma} \inf_{v \leq \zeta_u} |W_u(v)| = \hat{T}_0 = \inf\{|x| : x \in \mathcal{R}\} = r_0 - T_0 \quad Q_{x_0} - \text{a.s.}$$

Therefore (S.22) is equivalent to

$$\int_0^\sigma 1(\inf_{v \leq \zeta_u} |W_u(v)| = \hat{T}_0) \, du = 0 \quad Q_{x_0} - \text{a.s.}$$

The historical process, $(H_t, t \geq 0)$ is an inhomogeneous Markov process under $N_{x_0}$ taking values in $M_F(C(\mathbb{R}_+, \mathbb{R}^d))$—see [4] or p. 64 of [17] to see how it is easily defined from the snake $W$. The latter readily implies

$$\int_0^\infty H_t(\phi) \, dt = \int_0^\sigma \phi(W_u) \, du \quad \text{for all non-negative Borel } \phi,$$

where we have extended $W_u$ to $\mathbb{R}_+$ in the obvious manner. Recalling (7.1) and letting $X$ be the SBM under $N_{x_0}$ as usual, we have

$$\mathbb{N}_{x_0} \left( \int_0^\infty 1(\inf_{v \leq \zeta_u} |W_u(v)| = \hat{T}_0) \, du \right)$$

$$\leq \mathbb{N}_{x_0} \left( \int_0^\infty \int 1(\inf_{v \leq \zeta_u} |y_{\bar{t}}| = \hat{T}_0) H_t(dy) \, dt \right) \quad \text{(by (S.25))}$$

$$\leq \int_0^\infty \mathbb{N}_{x_0} \left( \int 1 \left( \int_0^\infty X_s(\{x : |x| < \inf_{t' \leq t} |y(t')|\}) \, ds = 0 \right) H_t(dy) \right) \, dt,$$

where in the last line we use (S.23) and $y(\cdot) = y(\cdot \wedge t)$ $H_t$-a.a. $y \forall t \geq 0 \mathbb{N}_{x_0}$-a.e. Below we will let $B$ denote a $d$-dimensional Brownian motion starting at $x_0$ under $P^B$, $m_t = \inf_{\nu \leq t} |B_{\nu}| = |B_{\tau}|$ (for some $\tau < t$), and $L^x$ be the local time of the SBM $X$ (at time infinity). Fix $t > 0$ and use the Palm measure formula for $H_t$ (e.g. Proposition 4.1.5 of [4]) to see that (cf. (7.22) in [20])

$$\mathbb{N}_{x_0} \left( \int 1 \left( \int_0^\infty X_s(\{x : |x| < \inf_{t' \leq t} |y(t')|\}) \, ds = 0 \right) H_t(dy) \right)$$

$$= E^B_{x_0} \left( \exp \left( - \int_0^t \int 1 \left( \int_0^\infty X_s(\{x : |x| < m_t\}) \, ds > 0 \right) d\mathbb{N}_{B_u} \, du \right) \right)$$

$$\leq E^B_{x_0} \left( \exp \left( - \int_0^t \mathbb{N}_{B_u}(L^{B_{\tau_t}} > 0) \, du \right) \right).$$
It follows from (1.13), (1.14) and \( P_{\delta_x}(L^y = 0) = \exp(-N_x(L^y > 0)) \) (see, e.g., (2.12) in [20]) that

\[ N_x(L^y > 0) = 2(4 - d)|x - y|^{-2}. \]

Use this to bound (S.28) by

\[ E^{B_{10}}_{x0} \left( \exp\left(-\int_0^t \frac{2(4 - d)}{|B_s - B_{\tau_t}|^2} ds\right) \right). \]

A simple application of Lévy’s modulus for \( B \) shows the above integral is infinite a.s. and so proves that (S.26) equals zero. This implies (S.24), as required.

REFERENCES


