Proposition 9.1 Assume X is a r.v. on a countable sample space S (ie, X is a function from S to $\mathbb{R}$)

Then $E(X) = \sum_{w \in S} X(w) P(\{w\})$.

Note: The sum is a countable sum, hence a series, because S is countable. Assume S is finite the first time you read this. $P(\{w\})$ is the probability of the particular outcome w.

Proof Let $S_i = \{X = x_i\}$ where $x_1 \ldots x_n$ are the distinct possible values of X. (If X can take on countably many values, the proof is readily adapted by considering an ∞ sequence $x_1, x_2, \ldots$.)

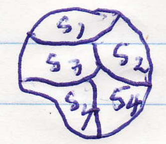
$$S_i = \{w : X(w) = x_i\} = \bigcup_{w \in S_i} \{w\} \quad \text{(a countable disjoint union)}$$

$$(1) \therefore P(S_i) = \sum_{w \in S_i} P(\{w\}) \quad \text{by countable additivity of } P$$

By definition: $E(X) = \sum_{i=1}^{n} x_i P(S_i)$

$$= \sum_{i=1}^{n} x_i \sum_{w \in S_i} P(\{w\}) \quad \text{by (1)}$$

S



$$= \sum_{i=1}^{n} \sum_{w \in S_i} x_i P(\{w\})$$

$$= \sum_{i=1}^{n} \sum_{w \in S_i} X(w) P(\{w\}) \quad \because \text{for } w \in S_i,\ X(w) = x_i$$

$$= \sum_{w \in S} X(w) P(\{w\})$$

The last line holds since $S_1 \cup \ldots \cup S_n$ partitions S into n disjoint sets and so summing w over each S_i is just summing w over S.