Nov. 4/11

<u>Theorem.</u> Assume X and Y are jointly continuous with joint p.d.f. $f(x,y)$. Then

(1) X and Y are independent

$\Leftrightarrow$ (2) $f(x,y) = f_X(x)\, f_Y(y)$

$\Leftrightarrow$ (3) $f(x,y) = g(x)\, h(y)$ for some functions $g(x)$ and $h(y)$.

<u>Proof.</u> (1) $\Rightarrow$ (2) Suppose X and Y are independent.

Recall from Nov. 2 Lecture that f is the joint pdf of X and Y iff

(*) $P((X,Y) \in I_1 \times I_2) = \iint_{I_1 \times I_2} f(x,y)\,dx\,dy$ for all <u>intervals</u> I_1, I_2

For intervals I_1, I_2 we have

$$
\begin{aligned}
P((X,Y) \in I_1 \times I_2) &= P(X \in I_1, Y \in I_2) \\
&= P(X \in I_1)\, P(Y \in I_2) \quad \text{(by independence)} \\
&= \int_{I_1} f_X(x)\,dx \int_{I_2} f_Y(y)\,dy \\
&= \int_{I_1} f_X(x) \left[\int_{I_2} f_Y(y)\,dy \right] dx \\
&= \iint_{I_1 \times I_2} f_X(x) f_Y(y)\,dx\,dy.
\end{aligned}
$$

So we may apply (*) to conclude that (X,Y) has joint pdf
$f(x,y) = f_X(x) f_Y(y)$.

(2) $\Rightarrow$ (1) Suppose $f(x,y) = f_X(x)\, f_Y(y)$. Let $I, J \subset \mathbb{R}$.

$$
\begin{aligned}
P(X \in I, Y \in J) &= P((X,Y) \in I \times J) \\
&= \iint_{I \times J} f_X(x)\, f_Y(y)\,dx\,dy \\
&= \int_I \int_J f_X(x)\, f_Y(y)\,dx\,dy = \int_I f_X(x)\,dx \times \int_J f_Y(y)\,dy \\
&= \underline{P(X \in I)\, P(Y \in J)}
\end{aligned}
$$

$\therefore$ X and Y are independent

(2) $\Rightarrow$ (3) Obvious

(3) $\Rightarrow$ (2) Suppose $f(x,y) = g(x)\,h(y)$ for some functions g, h.

(i) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{-\infty}^{\infty} g(x)\,h(y)\,dy = g(x)\int_{-\infty}^{\infty} h(y)\,dy \equiv g(x)\,C_h$.

Similarly $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{-\infty}^{\infty} g(x)\,h(y)\,dx = h(y)\int_{-\infty}^{\infty} g(x)\,dx$

(ii) $= h(y)\,C_g$

Integrate (i): $\overset{(*)}{1} = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{\infty} g(x)\,dx\,C_h \equiv C_g C_h$.

So (i) and (ii) imply:

$f(x,y) = g(x)\,h(y) = \dfrac{f_X(x)\,f_Y(y)}{C_h C_g} = f_X(x)\,f_Y(y)$, by (*)

since $g(x) = \dfrac{f_X(x)}{C_h}$

and $h(y) = \dfrac{f_Y(y)}{C_g}$