Lemma 1.23 Assume (IR) X is irreducible, recurrent.

Let $k \in S$. Let π be a stationary measure for X such that $\pi_k = 1$. Then $\pi_j = \rho_j(k)$ $\forall j \in S$.

(Recall $\rho_i(k) = E_k\left(\sum_{n=1}^{T_k} 1(X_n = i)\right)$, $i \in S$, is a stationary measure for X.)

Proof (1) $\pi_j = \sum_{i_1 \in S} \pi_{i_1} p_{i_1 j} = \sum_{i_1 \neq k} \pi_{i_1} p_{i_1 j} + p_{kj}$

Apply (1) to each π_{i_1} in the above:

$$\pi_j = \sum_{i_1 \neq k} \left[\sum_{i_2 \neq k} \pi_{i_2} p_{i_2 i_1} + p_{k i_1}\right] p_{i_1 j} + p_{kj}$$

$$= \sum_{i_1 \neq k} \sum_{i_2 \neq k} \pi_{i_2} p_{i_2 i_1} p_{i_1 j} + \sum_{i_1 \neq k} p_{k i_1} p_{i_1 j} + p_{kj}$$

Iterate this procedure $n-2$ more times:

$$\pi_j = \sum_{i_1 \neq k} \cdots \sum_{i_n \neq k} \pi_{i_n} p_{i_n i_{n-1}} \cdots p_{i_2 i_1} p_{i_1 j} + \sum_{i_1 \neq k} \cdots \sum_{i_{n-1} \neq k} p_{k i_{n-1}} p_{i_{n-1} i_{n-2}} \cdots p_{i_1 j}$$

$$+ \cdots + \sum_{i_1 \neq k} p_{k i_1} p_{i_1 j} + p_{kj}$$

Drop the 1st summation and use the Markov property on the other sums:

$$\pi_j \geq P_k(X_n = j, T_k \geq n) + P_k(X_{n-1} = j, T_k \geq n-1) + \cdots + P_k(X_2 = j, T_k \geq 2)$$
$$+ P_k(X_1 = j, T_k \geq 1)$$

$$= E_k\left(\sum_{m=1}^{n} 1(T_k \geq m, X_m = j)\right)$$

$$\xrightarrow{n \to \infty} E_k\left(\sum_{m=1}^{\infty} 1(T_k \geq m, X_m = j)\right) \quad \text{by (MCT)}$$

$$= \rho_j(k).$$

$\therefore \pi_j \geq p_j(k) \quad \forall j \in S.$

$\therefore \mu(j) = \pi_j - p_j(k) \geq 0$ is a s.m. s.t. $\mu_k = 0$.

As $j \to k$, $\exists n \in \mathbb{N}$ s.t. $p_{jk}(n) > 0$.

$$\therefore 0 = \mu_k = \sum_i \mu_i p_{ik}(n) \geq \mu_j \underbrace{p_{jk}(n)}_{> 0}$$

$\therefore \underline{\mu_j = 0}$ for any $j \in \mathbb{N}$. ie $\underline{\pi_j = p_j(k) \ \forall j \in S}$ $\square$