LONG-TIME BEHAVIOUR AND CO-EXISTENCE IN A MUTUALLY CATALYTIC BRANCHING MODEL

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Summary

We study a system of two interacting populations which undergo random migration and mutually catalytic branching. The branching rate of one population at a site is proportional to the mass of the other population at the site. The system is modelled by an infinite system of stochastic differential equations, allowing symmetric Markov migration, if the set of sites is discrete ($\mathbb{Z}^d$), or by a stochastic partial differential equation with Brownian migration if the set of sites is the real line. A duality technique of Leonid Mytnik, which gives uniqueness in law, is used to examine the long-time behaviour of the solutions. For example, with uniform initial conditions, the process converges to an equilibrium distribution as $t \to \infty$, and there is co-existence of types in the equilibrium "iff" the random migration is transient.

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[1] Research supported by an NSERC Research Grant and Collaborative Grant.

AMS 1991 *subject classifications.* Primary 60G57, 60H15, 60J80; Secondary 60H10, 60J60, 60K35.

Key words and phrases. Measure-valued processes, branching, superprocesses, equilibrium distribution, dual process, co-existence of types.

Running title: Mutually catalytic branching

1. Introduction and Statement of Results.

There has been considerable recent interest in the study of branching measure-valued diffusions or superprocesses for which branching can occur only in the presence of a (random or deterministic) catalytic medium. One interesting feature of this work is that in a variety of settings, branching in a singular medium leads to absolutely continuous measure-valued processes (see Dawson and Fleischmann (1994, 1995) or Delmas (1996)) in higher dimensions.

Another important development is the study of interactive models based on superprocesses or Fleming-Viot type processes (e.g. see Perkins (1995) or Dawson and March (1995)). Both these references characterize interactive models in which the branching or resampling rate depends on the state of the system. In both cases, however, the basic uniqueness results are not as general as one would hope (for quite different reasons).

In this work we study a class of stochastic models proposed by Carl Mueller which exhibit interactive, "mutually catalytic" branching and are therefore closely connected to both of the developments described above. There are two types of particles (you may call them male and female although the biology implied by the mathematical model will be highly suspect) each of which may branch only when the other is present. More precisely, the branching rate of each type at a site is proportional to the amount of the other type present at that site. If the set of sites is the real line, $\mathbb{R}$, this leads to the following system of stochastic partial differential equations in which $\gamma > 0$ and $\dot{W}_i(t,x)$ $(i = 1,2)$ are independent space-time white noises on $\mathbb{R}_+ \times \mathbb{R}$:

$$(SPDE)_{u_0,v_0} \qquad \begin{aligned} &(i)\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t,x) + (\gamma u(t,x)v(t,x))^{1/2}\dot{W}_1(t,x); \quad u(0,x) = u_0(x) \\ &(ii)\frac{\partial v}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2 v}{\partial x^2}(t,x) + (\gamma u(t,x)v(t,x))^{1/2}\dot{W}_2(t,x); \quad v(0,x) = v_0(x). \end{aligned}$$

The precise meaning of these equations (involving smooth test functions) is recalled below. In the simple one-dimensional catalytic superprocess, one has (i) where v is the density of a given random or deterministic catalyst which is not affected by u. The interactive nature of (i) and (ii) invalidates the techniques used to study these catalytic models and forces the development of new methods. Our hope was that due to the simple and symmetric nature of this model (linear branching rates) the general uniqueness difficulties encountered in the work on general interactive branching models could be resolved in this case. This problem was recently solved by Leonid Mytnik (1996) who proved weak uniqueness of solutions to $(SPDE)_{u_0,v_0}$ in his Ph.D. thesis.

Another interesting feature of $(SPDE)_{u_0,v_0}$ is its potential extension to higher dimensions. Here it may be formulated as a measure-valued martingale problem involving the collision local time of the two types (see Barlow, Evans and Perkins (1991)). This change is necessary because it seems unlikely that the solutions would be absolutely continuous in more than one spatial dimension. Note that if u was an absolutely continuous measure, then the results for superprocesses (Dawson and Hochberg (1979), Perkins (1988)) suggest that v would be a singular measure. If u was a singular measure, then the results of Dawson and Fleischmann (1995) and

Delmas (1996), cited above, suggest that v should be absolutely continuous. The same reasoning holds when the roles of u and v are interchanged and we are led to conclude that both u and v have non-trivial absolutely continuous and singular parts. Although the situation looks intriguing, the existence of solutions in higher dimensions is unresolved.

Once the basic issues of existence and uniqueness are resolved, it is natural to ask about the long term behaviour of solutions to $(SPDE)$. Let m denote Lebesgue measure, $u, v \in (0, \infty)$ and set $(u_0, v_0) = (um, vm)$. Dawson and Fleischmann (1988, 1995) have shown that for the catalytic model in which $v(t, x)$ is the density of a one-dimensional super-Brownian motion and $u(t, x)$ satisfies $(SPDE)_{u_0,v_0}$ (i), as $t \to \infty$ v_t becomes extinct in bounded regions a.s. and u_t approaches um in probability, in the vague topology. A naive guess for solutions of $(SPDE)_{um,vm}$ is that as $t \to \infty$, (u_t, v_t) converges to a mixture of $(0, vm)$ and $(um, 0)$. We show that as $t \to \infty$, (u_t, v_t) converges <u>weakly</u> to $(u_\infty m, v_\infty m)$, where $u_\infty v_\infty = 0$, u_∞ has mean u, and v_∞ has mean v, and give a simple explicit description of the law of (u_∞, v_∞) (Theorem 1.8). Although this shows that only one type survives in the limit, the convergence is only weak and so it may well be that the "dominant type" changes infinitely often as $t \to \infty$. This remains open.

We suspect, as for super-Brownian motion (see Dawson (1977)), that the asymptotic behaviour of solutions is quite different in the transient case, $d \geq 3$ (assuming that such solutions exist). As the existence of solutions in higher dimensions is unresolved it is natural to study the limiting behaviour of solutions in transient and recurrent settings by considering the analogues of $(SPDE)$ on the lattice $\mathbb{Z}^d$, in which Brownian motion on the line is replaced by a symmetric Markov chain on $\mathbb{Z}^d$. We now describe our results in some detail in this discrete setting.

Let $\big(\xi_t, (P^k : k \in \mathbb{Z}^d)\big)$ be a continuous time $\mathbb{Z}^d$-valued Markov chain and set $p_t(j, k) = P^j(\xi_t = k)$. Let $Q = (q_{jk})$ denote the associated Q-matrix, that is, q_{jk} is the jump rate from j to k $(j \neq k)$ and $q_{jj} = -\sum_{k \neq j} q_{jk} > -\infty$. If $k = (k_1, \dots k_d) \in \mathbb{Z}^d$, let $|k| = \sum_{i=1}^{d} |k_i|$. We assume:

(H_0) $\|q\|_\infty = \sup_j |q_{jj}| < \infty$

(H_1) For each $j, k \in \mathbb{Z}^d$, $q_{jk} = q_{kj}$ and hence $p_t(j, k) = p_t(k, j)$ for all $t \geq 0$.

(H_2) There are increasing, positive functions $c_{1.1}(T, \lambda)$ and $\lambda'(\lambda)$ on $\mathbb{R}_+^2$ and $\mathbb{R}_+$, respectively, such that $\lim_{\lambda \downarrow 0} \lambda'(\lambda) = 0$ and for any T, $\lambda > 0$,

$$\sum_k \left(|q_{jk}| + p_t(j, k)\right) e^{\lambda |k|} \leq c_{1.1}(T, \lambda) e^{\lambda'(\lambda)|j|} \qquad \forall t \in [0, T], j \in \mathbb{Z}^d.$$

It is easy to check these hypotheses for continuous time symmetric random walks with a sub-exponential tail (Lemma 2.1), including of course simple symmetric random walk on $\mathbb{Z}^d$ for which $q_{j\, j \pm e_i} = (2d)^{-1}$ (e_i is the i^{th} element in the standard unit basis), $q_{jj} = -1$ and $q_{jk} = 0$, otherwise.

If $\gamma > 0$, the discrete analogue of $(SPDE)_{u_0,v_0}$ is the following system of stochastic differential equations:

$$(LS)_{u_0,v_0} \quad \begin{aligned} u_t(k) &= u_0(k) + \int_0^t u_s Q(k)\,ds + \int_0^t \big(\gamma u_s(k) v_s(k)\big)^{1/2} dB_s^k, t \geq 0, k \in \mathbb{Z}^d \\ v_t(k) &= v_0(k) + \int_0^t v_s Q(k)\,ds + \int_0^t \big(\gamma u_s(k) v_s(k)\big)^{1/2} dW_s^k, t \geq 0, k \in \mathbb{Z}^d. \end{aligned}$$

Here, $u_s Q(k) = \sum_j u_s(j) q_{jk}$ and $u_0 : \mathbb{Z}^d \to \mathbb{R}_+$, $v_0 : \mathbb{Z}^d \to \mathbb{R}_+$ are a pair of given initial conditions.

We say that (u, v, B, W) is a solution of $(LS)_{u_0,v_0}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ iff

(i) $\{B_\cdot^k, W_\cdot^k : k \in \mathbb{Z}^d\}$ are independent one-dimensional $\mathcal{F}_t$-Brownian motions

(ii) $\sum_j |u_s(j) q_{jk}| < \infty \quad \forall s \geq 0, k \in \mathbb{Z}^d$ $\mathbb{P}$-a.s.

(iii) $\{u(k), v(k) : k \in \mathbb{Z}^d\}$ are continuous, non-negative $\mathcal{F}_t$-adapted stochastic processes satisfying $(LS)_{u_0,v_0}$ for all $t \geq 0$, $k \in \mathbb{Z}^d$ a.s.

We stress that in all of our results on (LS) the hypotheses (H_0)–(H_2) will be in force.

Notation. If $\lambda \in \mathbb{R}$, $\phi_\lambda(k) = e^{\lambda|k|}$ for $k \in \mathbb{Z}^d$, and we will use the same notation to denote this function on $\mathbb{R}^d$. If $u, v : \mathbb{Z}^d \to \mathbb{R}$, $\langle u, v\rangle = \sum_k u(k)v(k)$, providing this series is absolutely convergent or $u \geq 0$ and $v \geq 0$. If $\lambda \in \mathbb{R}$ and $u, v : \mathbb{Z}^d \to \mathbb{R}$, let $|u - v|_\lambda = \langle |u - v|, \phi_\lambda\rangle$.

An appropriate state space for solutions of $(LS)_{u_0,v_0}$ will be $M_{tem} \times M_{tem}$, where

$$M_{tem} = M_{tem}(\mathbb{Z}^d) = \{u : \mathbb{Z}^d \to \mathbb{R}_+ : \langle u, \phi_\lambda\rangle < \infty \qquad \forall \lambda < 0\}.$$

We topologize M_{tem} by the metric

$$d_{tem}(u, v) = \sum_{n=1}^{\infty} 2^{-n} \left(|u - v|_{-\lambda_n} \wedge 1\right)$$

where $\lambda_n \downarrow 0$. Let $\Omega_{tem} = C(\mathbb{R}_+, M_{tem} \times M_{tem})$, equipped with the compact-open topology.

Theorem 1.1. Let $u_0, v_0 \in M_{tem}$.

(a) There is a solution (u, v, B, W) of $(LS)_{u_0,v_0}$ such that $(u, v) \in \Omega_{tem}$ a.s.

(b) Any solution (u, v) of $(LS)_{u_0,v_0}$ has sample paths in Ω_{tem} and the laws of any two solutions of $(LS)_{u_0,v_0}$ on Ω_{tem} coincide.

The existence of a solution follows from standard techniques (e.g. Shiga and Shimizu (1980)). Several existence results and properties of solutions are presented in Section 2 (the existence part of Theorem 1.1 follows from Theorem 2.2) but most of the proofs are relegated to the Appendix. The uniqueness was obtained by Mytnik (1996) through an elegant duality argument. We present a proof in Section 2 for completeness (see Theorem 2.4) because our setting is slightly different (Mytnik worked with $(SPDE)_{u_0,v_0}$ and integrable u_0 and v_0 but the same ideas work here) and the duality relation will be our main tool in studying the long time behaviour of solutions to (LS). The strong Markov and Feller properties of the solution will follow easily (Corollary 2.7). Let $\mathbb{P}_{u_0,v_0}$ denote the law of the solution to $(LS)_{u_0,v_0}$ on Ω_{tem} and let (u_t, v_t) denote the coordinate variables on this space.

As with bilinear systems of s.d.e.'s (e.g. Gauthier (1996)) it is not hard to use Itô's lemma to see that N^{th} order moments of the form $\mathbb{P}\left(\prod_{i=1}^{p} u(t,k_i)^{n_i} v(t,k_i)^{m_i}\right)$ where $\sum_{i=1}^{p} m_i + n_i \le N$, $p \in \mathbb{N}$, $k_i \in \mathbb{Z}^d$, solve a system of uniquely solvable differential equations. In fact one may explicitly describe the above moments in terms of a dual system of N particles in $\mathbb{Z}^d$ which are of two possible colours, migrate as copies of ξ_t, and can change colour in an appropriate manner. Unfortunately the moments grow too quickly for the moment problem to be well-posed and so this approach does not establish uniqueness in law of solutions to (LS). (The same situation holds for solutions of $(SPDE)$.) We will use these moments for $N \le 2$ (they are derived in Theorem 2) but the expressions for moments with $N \ge 4$ are rather complex and so the above dual process will not be described here.

Assume first that $\langle u_0, 1\rangle + \langle v_0, 1\rangle < \infty$. Then it is not hard to see that $\langle u_t, 1\rangle$ and $\langle v_t, 1\rangle$ are non-negative martingales under $\mathbb{P}_{u_0,v_0}$ (Theorem 2.2(d)) and hence converge $\mathbb{P}_{u_0,v_0}$-a.s. as $t \to \infty$ by the Martingale Convergence Theorem. Let $\langle u_\infty, 1\rangle$ and $\langle v_\infty, 1\rangle$ denote their a.s. limits. We say co-existence of types is possible if $\mathbb{P}_{u_0,v_0}(\langle u_\infty, 1\rangle\langle v_\infty, 1\rangle > 0) > 0$ whenever $0 < \langle u_0, 1\rangle < \infty$ and $0 < \langle v_0, 1\rangle < \infty$. Co-existence of types is impossible if $\mathbb{P}_{u_0,v_0}(\langle u_\infty, 1\rangle\langle v_\infty, 1\rangle > 0) = 0$ whenever $\langle v_0, 1\rangle + \langle v_0, 1\rangle < \infty$.

Notation. $g_t(j,k) = \int_0^t p_s(j,k)\, ds, \qquad t \in [0,\infty], \quad j,k \in \mathbb{Z}^d.$

The following result states that under appropriate "homogeneity" conditions, co-existence of types is possible if and only if ξ_t is transient.

Theorem 1.2. Assume $\langle u_0, 1\rangle + \langle v_0, 1\rangle < \infty$.

(a) If $\sup_k g_\infty(k,k) < \infty$, then co-existence of types is possible.

(b) Assume $\exists c_{1.2} > 0$ such that $\forall j \in \mathbb{Z}^d$ $\exists T_{1.2}(j)$ such that

$$g_T(j,j) \ge c_{1.2} \sup_k g_T(k,k) \quad \forall T > T_{1.2}(j). \tag{1.1}$$

If $P_j(\xi_t = k$ for some $t > 0) = 1$ $\forall j,\ k \in \mathbb{Z}^d$, then co-existence of types is impossible.

The result is proved in Section 3. Under our hypothesis (H_0), the assumption in (a) is equivalent to uniform transience of the chain, i.e., $\sup_k P_k(\xi_t = k$ for arbitrarily large $t) < 1$. The hypotheses of (b) simply state that all states communicate, are recurrent and are comparable in the sense of (1.1). We suspect these conditions are not optimal. (a) is easy to prove through a first moment argument. (b) is more involved and relies on an integral equation for $u_t(k)$ and

$v_t(k)$ (the mild form of (LS)) involving Green's functions. The proof does not use the uniqueness of solutions to (LS) or duality.

Corollary 1.3. Let (ξ_t) be simple symmetric random walk on $\mathbb{Z}^d$. Co-existence of types is possible if $d \geq 3$ and impossible if $d \leq 2$.

Proof. Let $\{S_n : n \in \mathbb{Z}_+\}$ be the discrete time simple symmetric random walk on $\mathbb{Z}^d$. The holding times of (ξ_t) are mean one exponential variables and so

$$g_\infty(k,k) = g_\infty(0,0) = \sum_{n=0}^{\infty} P_0(S_n = 0)$$

which is finite for $d > 2$. Theorem 1.2(a) gives the first assertion. (1.1) is trivially satisfied by translation invariance and so Theorem 1.2 (b) and the point recurrence of random walk for $d \leq 2$ gives the second assertion. $\blacksquare$

Even when co-existence of types is impossible, it is not clear from the above if extinction of one type can occur in finite time. A related question is whether or not we may take the probability of co-existence to be one when co-existence is possible. In a companion article (Mueller and Perkins (1997)) it will be shown that irregardless of the recurrence or transience properties of (ξ_t), one may select initial conditions so that finite-time extinction of one type occurs with probability zero and other (non-zero) initial conditions for which this probability is arbitrarily close to one. In particular, this shows that in the transient case, $P_{u_0,v_0}(\langle u_\infty, 1\rangle\langle v_\infty, 1\rangle > 0)$ may be arbitrarily close to zero for certain non-zero u_0 and v_0.

Finally, in any case one has $\langle u_\infty, 1\rangle > 0$ or $\langle v_\infty, 1\rangle > 0$ P_{u_0,v_0}-a.s. for $u_0 + v_0 \neq 0$ because Theorem 2.2 (d) allows one to see that $(\langle u_t, 1\rangle, \langle v_t, 1\rangle)$ is the time change of a planar Brownian motion stopped when it exits the first quadrant and hence is bounded away from $(0,0)$ a.s.

Next we turn to the setting when $u_0(k) = u$ and $v_0(k) = v$ for all $k \in \mathbb{Z}^d$ for some $u, v \geq 0$. Write $\mathbb{P}_{u,v}$ for the law of (u,v) with these initial conditions.

Theorem 1.4. For each $u, v \geq 0$, $\mathbb{P}_{u,v}\big((u_t, v_t) \in \cdot\big)$ converges weakly on $(M_{tem})^2$ as $t \to \infty$ to a stationary initial distribution $\mathbb{P}_{u,v}\big((u_\infty, v_\infty) \in \cdot\big)$ for solutions of (LS). Moreover for each $k \in \mathbb{Z}^d$ $\mathbb{P}_{u,v}\big(u_\infty(k)\big) = u$ and $\mathbb{P}_{u,v}\big(v_\infty(k)\big) = v$. If $q_{j,k} = \tilde{q}(j-k)$ for all $j, k \in \mathbb{Z}^d$, then for each $k \in \mathbb{Z}^d$, $\mathbb{P}_{u,v}\left((u_\infty(k+\cdot), v_\infty(k+\cdot)) \in \cdot\right) = \mathbb{P}_{u,v}\left((u_\infty, v_\infty) \in \cdot\right)$ (i.e. $\mathbb{P}_{u,v}\left((u_\infty, v_\infty) \in \cdot\right)$ is a stationary random field on $\mathbb{Z}^d$).

The existence of an equilibrium distribution will be an easy consequence of Mytnik's duality relation and so the above result is proved in Section 2. This duality relation relies on the fact that $(u+v, u-v)$ is self dual with respect an appropriate class of functionals. It would appear that this self-referential technique would not shed much light on the nature of the equilibrium found in Theorem 1.4 but the duality describes the above equilibrium in terms of $(\langle u_\infty, 1\rangle, \langle v_\infty, 1\rangle)$ for initial conditions with finite total mass. Therefore we may use Theorem 1.2 to derive the following results in Section 4. Together they state that under appropriate regularity conditions, co-existence of types in equilibrium is possible if ξ_t is transient and impossible if ξ_t is recurrent.

Notation. If $x \in \mathbb{R}_+$, $\overline{x} : \mathbb{Z}^d \to \mathbb{R}_+$ is the map which is constant and equal to x.

Theorem 1.5. Assume ξ_t satisfies the (recurrence) hypotheses of Theorem 1.2(b) (including (1.1)). Let $u, v \geq 0$ and assume $B_t = (B_t^1, B_t^2)$ is a planar Brownian motion starting at (u,v)

under $P^0_{u,v}$. If $T = \inf\{t : B^1_t B^2_t = 0\}$, then

$$\mathbb{P}_{u,v}\big((u_\infty, v_\infty) \in \cdot\big) = P^0_{u,v}\left((\overline{B^1_T}, \overline{B^2_T}) \in \cdot\right).$$

In particular, $u_\infty = 0$ or $v_\infty = 0$ $\mathbb{P}_{u,v}$-a.s.

Remark. One readily calculates that

$$\begin{aligned} P^0_{u,v}(B^1_T \in dx, B^2_T = 0) &= P^0_{v,u}(B^2_T \in dx, B^1_T = 0) \\ &= 4uv\pi^{-1}x\left(4u^2v^2 + (x^2+v^2-u^2)^2\right)^{-1} dx \quad x > 0. \end{aligned}$$

For example, apply the conformal mapping $z \to \sqrt{z}$ to the Cauchy distribution obtained as the exit distribution of Brownian motion from the upper half-plane.

Theorem 1.6. Assume all states are transient for ξ, i.e., $g_\infty(k,k) < \infty$ for all $k \in \mathbb{Z}^d$. Let u, $v > 0$.

(a) $u_\infty(k) > 0$ and $v_\infty(k) > 0 \quad \forall k \in \mathbb{Z}^d \quad \mathbb{P}_{u,v}$-a.s.

(b) $Cov\big(u_\infty(j), u_\infty(k)\big) = Cov(v_\infty(j), v_\infty(k)\big) = \frac{\gamma uv}{2} g_\infty(j,k)$ and $Cov\big(u_\infty(j), v_\infty(k)\big) = 0$ $\forall j, k \in \mathbb{Z}^d$.

In Section 5 an elementary ergodic theorem is proved under the transience hypotheses of Theorem 1.6. The pathwise behaviour of (u_t, v_t) for large t is not well understood in the recurrent case. Some partial results and open questions are discussed in Section 5 when ξ_t is a simple symmetric random walk on $\mathbb{Z}$ or $\mathbb{Z}^2$.

We return now to $(SPDE)_{u_0,v_0}$ and introduce the analagous state space for solutions. At times we adopt the notation used in the lattice case but there should be no confusion as the context is quite different.

Notation. $C(\mathbb{R})$ is the space of continuous real-valued functions on the real line with the compact-open topology; $C_c(\mathbb{R})$ is the subspace of functions with compact support. If $\lambda \in \mathbb{R}$ and $g \in C(\mathbb{R})$, let $|g|_\lambda = \sup\limits_{x\in\mathbb{R}} e^{\lambda|x|}|g(x)|$. $C_{tem} = C_{tem}(\mathbb{R}) = \{f \in C(\mathbb{R}) : |f|_\lambda < \infty \quad \forall \lambda < 0\}$ and we topologize C_{tem} by the metric $d_{tem}(f,g) = \sum\limits_{n=1}^{\infty} 2^{-n}(|f-g|_{-\lambda_n} \wedge 1)$ where $\lambda_n \downarrow 0$. Let $C^2_{tem} = \{f \in C_{tem} : f' \in C$ and $f'' \in C_{tem}\}$ and $C^+_{tem} = \{f \in C_{tem} : f \geq 0\}$. We use these superscripts with other classes of functions (e.g. C^2_c, $C^+(\mathbb{R})$) without further explanation. Finally Ω_{tem} is the space of $\left(C^+_{tem}\right)^2$-valued paths on $\mathbb{R}_+$ with the compact-open topology. There should be no confusion with our earlier (discrete space) definition of Ω_{tem}.

If f, $g \in C(\mathbb{R})$, let $\langle f, g\rangle = \int\limits_{\mathbb{R}} f(x)g(x)\,dx$ whenever the integrand is integrable or non-negative.

Definition. (u, v, W_1, W_2) is a solution of $(SPDE)_{u_0,v_0}$ on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ iff

(i) W_1, W_2 are independent $(\mathcal{F}_t)$-adapted white noises on $\mathbb{R}_+ \times \mathbb{R}$ (see Ch. 3 of Walsh (1986)).

(ii) $\{(u(t,\cdot),v(t,\cdot)) : t \geq 0\}$ is a continuous $C^+_{tem}(\mathbb{R})^2$-valued, $(\mathcal{F}_t)$-adapted process such that $u(0,\cdot) = u_0(\cdot)$, $v(0,\cdot) = v_0(\cdot)$ a.s. and $\forall \phi \in C^2_c$

$$\langle u(t),\phi\rangle = \langle u_0,\phi\rangle + \int_0^t \langle u(s), \frac{\phi''}{2} > ds + \int_0^t \int (\gamma u(s,x)v(s,x))^{1/2}\phi(x)dW_1(s,x)$$

(SMR)

$$\langle v(t),\phi\rangle = \langle v_0,\phi\rangle + \int_0^t \langle v(s), \frac{\phi''}{2} > ds + \int_0^t \int (\gamma u(s,x)v(s,x))^{1/2}\phi(x)dW_2(s,x)$$

$$\forall t \geq 0 \quad \mathbb{P}\text{-a.s.}$$

Theorem 1.7. Let $u_0, v_0 \in C^+_{tem}$.

(a) There is a solution (u,v,W_1,W_2) of $(SPDE)_{u_0,v_0}$.

(b) Any two solutions of $(SPDE)_{u_0,v_0}$ have the same law on Ω_{tem} which we denote by $\mathbb{P}_{u_0,v_0}$.

Existence follows easily from standard arguments (e.g. Shiga (1994)) and is outlined in the Appendix (see Theorem 6.1). Uniqueness (see Theorem 6.3) was proved by Mytnik (1996) for $\langle u_0,1\rangle + \langle v_0,1\rangle < \infty$ and his duality argument easily extends to the above setting (and is similar to that given for (LS)). The proof is therefore omitted.

Co-existence of types is shown to be impossible when u_0, v_0 are rapidly decreasing at infinity (i.e. $\lim_{|x|\to\infty} \phi_\lambda(x)(u_0(x)+v_0(x)) = 0\ \forall \lambda > 0$) (see Theorem 6.6). The proof is similar to that of Theorem 1.2(b). As in the recurrent lattice case, this then leads to the asymptotic behaviour of the law of (u_t,v_t) as $t \to \infty$ with uniform initial conditions.

Notation. m denotes Lebesgue measure on $\mathbb{R}$,

$$M_{tem} = \{\mu : \mu \text{ a measure on } (\mathbb{R},\mathcal{B}(\mathbb{R})),\quad \langle \mu,\phi_\lambda\rangle \equiv \int \phi_\lambda d\mu < \infty \quad \forall \lambda < 0\}.$$

Let d_0 be a complete metric on the space of Radon measures on $\mathbb{R}$ inducing the vague topology, let $\lambda_n \downarrow 0$, and define a metric d on M_{tem} by

$$d(\mu,\nu) = d_0(\mu,\nu) + \sum_{n=1}^{\infty} (|\langle \mu,\phi_{-\lambda_n}\rangle - \langle \nu,\phi_{-\lambda_n}\rangle\,| \wedge 1)\, 2^{-n}$$

Then it is easy to see that (M_{tem},d) is a Polish space and $\mu_n \to \mu$ in M_{tem} iff $\lim_{n\to\infty}\langle \mu_n,\phi\rangle = \langle \mu,\phi\rangle \quad \forall \phi \in C_{\exp} = \{\phi \in C(\mathbb{R}) : |\phi|_\lambda < \infty \text{ for some } \lambda > 0\}$. C^+_{tem} may be viewed as a subset of M_{tem} by identifying $u(x)$ with the measure $A \to \int_A u dm$. Of course the induced topology is weaker than that on C^+_{tem}. If $u,v \geq 0$ we let $\mathbb{P}_{u,v}$ denote the law of the solution to $(SPDE)$ with $u_0(x) \equiv u$ and $v_0(x) \equiv v$.

Theorem 1.8. If B, T and $P^0_{u,v}$ are as in Theorem 1.5, then as $t \to \infty$

$$\mathbb{P}_{u,v}((u_t,v_t) \in \cdot) \xrightarrow{w} P^0_{u,v}((B^1_T m, B^2_T m) \in \cdot)$$

in the sense of weak convergence of probabilities on $(M_{tem})^2$.

This is proved in Section 6 along with the other results on $(SPDE)$. In many instances arguments here are only outlined as they are similar to the corresponding proofs in the lattice case. See the Remark following Theorem 1.5 for an explicit description of the above law.

2. Existence, Uniqueness and Basic Properties for the Discrete Model.

Lemma 2.1. Assume ξ_t is a continuous time random walk on $\mathbb{Z}^d$ with symmetric step distribution $F(x)$ satisfying $\int_{\mathbb{Z}^d} \phi_\lambda dF < \infty\ \forall\lambda > 0$. More precisely $q_{jk} = F(k-j)$ for $j \neq k$ in $\mathbb{Z}^d$. Then (H_0), (H_1) and (H_2) hold with $\lambda'(\lambda) = \lambda$.

Proof. Let $\Phi(\lambda) = \int \phi_\lambda dF$. Then

$$\begin{aligned}\sum_k \big(|q_{jk}| + p_t(j,k)\big)e^{\lambda|k|} &\leq e^{\lambda|j|}\left(\sum_k \big(F(k-j) + p_t(0,k-j)\big)e^{\lambda|k-j|}\right) + e^{\lambda|j|}\\ &= e^{\lambda|j|}\left(1 + \Phi(\lambda) + P_0(e^{\lambda|\xi_t|})\right).\end{aligned}$$

Now for $t \leq T$ we have

$$P_0\left(e^{\lambda|\xi_t|}\right) \leq \sum_{n=0}^{\infty} e^{-t}t^n\Phi(\lambda)^n/n! \leq \exp\big(T(\Phi(\lambda)-1)\big).$$

This proves (H_2) with $c_{1.1}(T,\lambda) = 1+\Phi(\lambda)+\exp\left(T\big(\Phi(\lambda)-1\big)\right)$. (H_0) and (H_1) are obvious. ∎

Notation.

$$\begin{aligned}M_F = M_F(\mathbb{Z}^d) &= \{u : \mathbb{Z}^d \to \mathbb{R}_+ : \sum_k u(k) < \infty\},\\ M_{rap} = M_{rap}(\mathbb{Z}^d) &= \{u : \mathbb{Z}^d \to \mathbb{R}_+ : \langle u, \phi_\lambda\rangle < \infty \quad \forall\lambda \in \mathbb{R}\},\\ M_b = M_b(\mathbb{Z}^d) &= \{u : \mathbb{Z}^d \to \mathbb{R}_+ : \sup_k u(k) < \infty\}.\end{aligned}$$

We use M^s_{tem} to denote those functions $u : \mathbb{Z}^d \to \mathbb{R}$ such that $|u| \in M_{tem}$ and similarly define M^s_F, M^s_{rap}, M^s_b. Topologize M_{rap} by choosing $\mu_n \uparrow \infty$ and using the metric $d_{rap}(u,v) = \sum_{n=1}^{\infty} 2^{-n}(|u-v|_{\mu_n} \wedge 1)$. M_F is topologized by the ℓ^1-norm, $\|u-v\|_1 = \sum_k |u(k)-v(k)|$ and M_b is topologized by the ℓ^∞-norm. These same metrics are extended to M^s_b, M^s_F, M^s_{rap} and M^s_{tem}. (Recall that d_{tem} was defined on M_{tem} in the previous section.)

Ω_{rap} is the space of $M_{rap} \times M_{rap}$-valued paths on $\mathbb{R}_+$ with the compact-open topology and Ω_F is the same space with M_F in place of M_{rap}.

Let $S_n = \mathbb{Z}^d \cap [-n,n]^d$ for $n \in \mathbb{N}$, $P_tf(j) = \sum_k p_t(j,k)f(k)$ is the semigroup associated with ξ_t, and $Qf(k) = \sum_j q_{kj}f(j)$ is its generator.

Theorem 2.2. (a) If $u_0, v_0 \in M_{tem}$, there is a solution (u, v, B, W) to $(LS)_{u_0,v_0}$ on some $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ such that (u, v) has sample paths in Ω_{tem} a.s.

(b) Let (u, v, B, W) be any solution to $(LS)_{u_0,v_0}$ on some $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, where $u_0, v_0 \in M_{tem}$.

(i) Then $(u_., v_.) \in \Omega_{tem}$ a.s. and $\forall \lambda,\ T > 0$, $\mathbb{P}\left(\sup_{t \le T} \langle u_t + v_t, \phi_{-\lambda}\rangle\right) < \infty$.

(ii) If $\phi,\ \psi \in M^s_{rap}$, then

$$\langle u_t, \phi\rangle = \langle u_0, P_t\phi\rangle + N^u_t(t, \phi), \qquad \langle v_t, \psi\rangle = \langle v_0, P_t\psi\rangle + N^v_t(t, \psi),$$

where

$$N^u_s(t, \phi) = \sum_k \int_0^s P_{t-r}\phi(k)\big(\gamma u_r(k) v_r(k)\big)^{1/2} dB^k_r, \quad s \le t$$

and

$$N^v_s(t, \psi) = \sum_k \int_0^s P_{t-r}\psi(k)\big(\gamma u_r(k) v_r(k)\big)^{1/2} dW^k_r, \quad s \le t$$

are orthogonal continuous square-integrable $\mathcal{F}_t$-martingales (the series converge in L^2 uniformly in $s \le t$) with square functions

$$\langle N^u(t, \phi)\rangle_s = \gamma \int_0^s \sum_k P_{t-r}\phi(k)^2 u_r(k) v_k(k)\, dr$$

and

$$\langle N^v(t, \psi)\rangle_s = \gamma \int_0^s \sum_k P_{t-r}\psi(k)^2 u_r(k) v_r(k)\, dr.$$

(iii) If $\phi, \psi : \mathbb{Z}^d \to \mathbb{R}_+$, then

$$\mathbb{P}(\langle u_t, \phi\rangle) = \langle u_0, P_t\phi\rangle, \quad \mathbb{P}(\langle v_t, \psi\rangle) = \langle v_0, P_t\psi\rangle, \qquad \text{and}$$
$$\mathbb{P}(\langle u_t, \phi\rangle\langle v_t, \psi\rangle) = \langle u_0, P_t\phi\rangle\langle v_0, P_t\psi\rangle.$$

(iv) Assume $\phi,\ \psi : \mathbb{Z}^d \to \mathbb{R}$ satisfy $|\phi(j)| + |\psi(j)| \le ce^{-\lambda|j|}\ \ \forall j \in \mathbb{Z}^d$ and some $c,\ \lambda > 0$. Then

$$\langle u_t, \phi\rangle = \langle u_0, \phi\rangle + \int_0^t \langle u_s, Q\phi\rangle\, ds + M^u_t(\phi)$$

$$\langle v_t, \psi\rangle = \langle v_0, \psi\rangle + \int_0^t \langle v_s, Q\psi\rangle\, ds + M^v_t(\psi)$$

where $\mathbb{P}(\int_0^t \langle u_s, |Q\phi|\rangle + \langle v_s, |Q\psi|\rangle\, ds) < \infty$, $M_t^u(\phi)$ and $M_t^v(\psi)$ are orthogonal square integrable $(\mathcal{F}_t)$-martingales such that $\langle M^u(\phi)\rangle_t = \int_0^t \gamma\langle u_s v_s, \phi^2\rangle\, ds$ and $\langle M^v(\psi)\rangle_t = \int_0^t \gamma\langle u_s v_s, \psi^2\rangle\, ds$.

(c) Let (u, v, B, W) be as in (b) but with $u_0, v_0 \in M_{rap}$. Then $(u, v) \in \Omega_{rap}$ a.s. and $\forall\lambda$, $T > 0$, $\mathbb{P}\left(\sup_{t\leq T}\langle u_t + v_t, \phi_\lambda\rangle\right) < \infty$. (b)(ii) holds for ϕ, $\psi \in M^s_{tem}$ and (b)(iv) holds if ϕ, $\psi : \mathbb{Z}^d \to \mathbb{R}$ satisfy $|\phi(j)| + |\psi(j)| \leq ce^{\lambda|j|}$ $\forall j \in \mathbb{Z}^d$ and some c, $\lambda > 0$.

(d) Let (u, v, B, W) be as in (b) but with $u_0, v_0 \in M_F$. Then $(u_\cdot, v_\cdot) \in \Omega_F$ a.s. and (b)(ii), (iv) hold for $\phi, \psi \in M^s_b$. In particular $\langle u_t, 1\rangle$ and $\langle v_t, 1\rangle$ are orthogonal square integrable continuous martingales with square function $\gamma \int_0^t \langle u_s, v_s\rangle\, ds$.

We give the proof in the Appendix. Although similar in spirit to arguments in Shiga and Shimizu (1980), the proof is necessarily different in some respects because of the different state spaces and several additional specialized properties.

Let (u_t, v_t) denote the coordinate mappings on Ω_{tem} and for clarity we use $(\tilde{u}_t, \tilde{v}_t)$ to denote the coordinate maps on Ω_{rap}. Following Mytnik (1996) we define

$$(X_t, Y_t) = (u_t + v_t, u_t - v_t), \qquad (\tilde{X}_t, \tilde{Y}_t) = (\tilde{u}_t + \tilde{v}_t, \tilde{u}_t - \tilde{v}_t).$$

State spaces for these processes are given by

$$E = \{(x, y) : x \in M_{tem}, y \in M^s_{tem}, |y(k)| \leq x(k)\ \forall k \in \mathbb{Z}^d\}$$

and

$$\tilde{E} = \{(x, y) \in E : x \in M_{rap}\} \supset E_f = \{(x, y) \in E : x \text{ has finite support}\}.$$

Define a metric d_E on E by $d_E\big((x, y), (x', y')\big) = d_{tem}(x, x') + d_{tem}(y, y')$ and similarly define $d_{\tilde{E}}$ on $\tilde{E}$, using d_{rap}. It is easy to check that (E, d_E) and $(\tilde{E}, d_{\tilde{E}})$ are Polish spaces.

If $(\phi, \psi) \in \tilde{E}$, define a continuous function $F_{\phi,\psi} : E \to \mathbb{C}$ by

$$F_{\phi,\psi}(x, y) = \exp\{-\langle x, \phi\rangle + i\langle y, \psi\rangle\}.$$

Lemma 2.3. (a) If $\lambda_n \downarrow 0$ and $M_n > 0$, then $K = \{(x, y) \in E : \langle x, \phi_{-\lambda_n}\rangle \leq M_n\ \forall n \in \mathbb{N}\}$ is compact in E.

(b) If P and Q are laws on E such that $P(F_{\phi,\psi}) = Q(F_{\phi,\psi})\ \forall(\phi, \psi) \in E_f$, then $P = Q$.

(c) Let $\{P_n\}$ be probabilities on E such that $\forall\lambda > 0\ \sup_n \int \langle x, \phi_{-\lambda}\rangle dP_n(x) = K_\lambda < \infty$ and $P_n(F_{\phi,\psi})$ converges as $n \to \infty$ $\forall(\phi, \psi) \in E_f$. Then $\{P_n\}$ converges weakly to a probability P_∞ on E and $\lim P_n(F_{\phi,\psi}) = P_\infty(F_{\phi,\psi})$ $\forall(\phi, \psi) \in E_f$.

Proof. (a) Let $\{(x_n, y_n)\}$ be in K. We will construct a subsequence converging to a point in K. We may assume $\lim_{n\to\infty} x_n(k) = x(k)$ and $\lim_{n\to\infty} y_n(k) = y(k)$ exist $\forall k \in \mathbb{Z}^d$ by a Cantor diagonalization argument. By Fatou's lemma $\langle x, \phi_{-\lambda_n}\rangle \le M_n$ and so $(x, y) \in K$. If $N \in \mathbb{N}$ and $0 < \lambda_{n_0} < \lambda/2$, then

$$\begin{aligned}\langle |x_n - x|, \phi_{-\lambda}\rangle &\le \langle |x_n - x|, 1_{S_N}\phi_{-\lambda}\rangle + e^{-\lambda N/2}\langle |x_n - x|, 1_{S_N^c}\phi_{-\lambda/2}\rangle \\ &\le \langle |x_n - x|, 1_{S_N}\phi_{-\lambda}\rangle + e^{-\lambda N/2}\sup_n\langle x_n + x, \phi_{-\lambda_{n_0}}\rangle \\ &\le \langle |x_n - x|, 1_{S_N}\phi_{-\lambda}\rangle + e^{-\lambda N/2}2M_{n_0}.\end{aligned}$$

This shows the left side approaches zero as $n \to \infty$ and the same is true of $\langle |y_n - y|, \phi_{-\lambda}\rangle$. It follows that $(x_n, y_n) \to (x, y)$ in E.

(b) Fix $N \in \mathbb{N}$ and let

$$E^{(N)} = \{(x, y) \in \mathbb{R}_+^{S_N} \times \mathbb{R}^{S_N} : |y(k)| \le x(k) \quad \forall k \in S_N\}.$$

For $(a, b) \in E^{(N)}$, define $F_{a,b} : E^{(N)} \to \mathbb{C}$ by $F_{a,b}(x, y) = \exp\{-\langle a, x\rangle + i\langle b, y\rangle\}$. If $\mathcal{A} = \{F_{a,b} : (a, b) \in E^{(N)}\}$, then the Stone-Weierstrass theorem shows that the complex linear span of $\mathcal{A}$ is dense in the space of continuous complex-valued functions on $E^{(N)}$ with limits at infinity. If P and Q are as in the statement of (b), it now follows easily that they have the same finite-dimensional distributions and therefore coincide.

(c) We first show that $\{P_n\}$ is tight, and hence relatively compact by Prohorov's theorem. Let $\epsilon > 0$, $\lambda_j \downarrow 0$ and choose $M_j \ge \epsilon^{-1}2^j K_{\lambda_j}$. Then $C_\epsilon = \{(x, y) \in E : \langle x, \phi_{-\lambda_j}\rangle \le M_j \quad \forall j \in \mathbb{N}\}$ is compact in E by (a), and

$$P_n(C_\epsilon^c) \le \sum_{j=1}^{\infty} \int \frac{\langle x, \phi_{-\lambda_j}\rangle dP_n(x)}{M_j} \le \sum_{j=1}^{\infty} K_{\lambda_j}/M_j \le \epsilon \qquad \forall n \in \mathbb{N},$$

thus proving tightness. The convergence of $P_n(F_{\phi,\psi})$ for $(\phi, \psi) \in E_f$ and (b) show that all limit points of $\{P_n\}$ coincide and hence $P_n \xrightarrow{w} P_\infty$ for some P_∞. The last assertion is then immediate by the continuity of $F_{\phi,\psi}$ on E. ∎

If $(\tilde{u}_0, \tilde{v}_0) \in M_{rap} \times M_{rap}$, then by Theorem 2.2 the law of a solution to $(LS)_{\tilde{u}_0,\tilde{v}_0}$ may be viewed as a probability on Ω_{tem} or Ω_{rap}. This convention is used in the following uniqueness theorem of L. Mytnik (1996).

Theorem 2.4. (a) If $(u_0, v_0) \in (M_{tem})^2$, there is a unique probability $\mathbb{P}_{u_0,v_0}$ on Ω_{tem} so that (u, v) has law $\mathbb{P}_{u_0,v_0}$ whenever (u, v, B, W) is a solution of $(LS)_{u_0,v_0}$ on some probability space.

(b) If $(u_0, v_0) \in (M_{tem})^2$, $(\tilde{u}_0, \tilde{v}_0) \in (M_{rap})^2$, then (recall the definitions of (X_t, Y_t), $(\tilde{X}_t, \tilde{Y}_t)$ prior to Lemma 2.3)

$$\mathbb{P}_{u_0,v_0}\left(\exp\{-\langle X_t, \tilde{X}_0\rangle + i\langle Y_t, \tilde{Y}_0\rangle\}\right) = \mathbb{P}_{\tilde{u}_0,\tilde{v}_0}\left(\exp\left\{-\langle X_0, \tilde{X}_t\rangle + i\langle Y_0, \tilde{Y}_t\rangle\right\}\right).$$

Proof. (b) Let $(\tilde{\phi}, \tilde{\psi}) \in \tilde{E}$ and write $F\big((\tilde{\phi}, \tilde{\psi}), (x, y)\big)$ for $F_{\tilde{\phi},\tilde{\psi}}(x, y)$. Define

$$AF(\tilde{\phi}, \tilde{\psi}, x, y) = F(\tilde{\phi}, \tilde{\psi}, x, y)\left[-\langle x, Q\tilde{\phi}\rangle + i\langle y, Q\tilde{\psi}\rangle + (\gamma^2/4)\langle x^2 - y^2, \tilde{\phi}^2 - \tilde{\psi}^2\rangle\right]. \tag{2.1}$$

Let $\mathbb{P}_{u_0,v_0}$ (respectively, $\mathbb{P}_{\tilde{u}_0,\tilde{v}_0}$) be the law of (u,v) where (u,v,B,W) is a solution of $(LS)_{u_0,v_0}$ (respectively, $(LS)_{\tilde{u}_0,\tilde{v}_0}$). Theorem 2.2(b)(iv) and Itô's Lemma show that ($\mathcal{F}_t$ is the canonical right-continuous filtration on Ω_{tem})

$$F(\tilde{\phi},\tilde{\psi},X_t,Y_t) - F(\tilde{\phi},\tilde{\psi},X_0,Y_0) - \int_0^t AF(\tilde{\phi},\tilde{\psi},X_s,Y_s)\,ds \equiv M_t(\tilde{\phi},\tilde{\psi}) \tag{2.2}$$

is a continuous $(\mathcal{F}_t)$-martingale under $\mathbb{P}_{u_0,v_0}$. If $(\phi,\psi)\in E$, Theorem 2.2(c), Itô's Lemma and the symmetry of Q show that

$$F(\tilde{X}_t,\tilde{Y}_t,\phi,\psi) - F(\tilde{X}_0,\tilde{Y}_0,\phi,\psi) - \int_0^t AF(\tilde{X}_s,\tilde{Y}_s,\phi,\psi)\,ds \equiv N_t(\phi,\psi) \tag{2.3}$$

is a continuous $(\mathcal{F}_t)$-martingale under $\mathbb{P}_{\tilde{u}_0,\tilde{v}_0}$. Let

$$f(s,t) = \mathbb{P}_{\tilde{u}_0,\tilde{v}_0}\times\mathbb{P}_{u_0,v_0}\left(F(\tilde{X}_s,\tilde{Y}_s,X_t,Y_t)\right).$$

Then (2.2) shows that

$$\begin{aligned} f(s,t) &= \mathbb{P}_{\tilde{u}_0,\tilde{v}_0}\left(F(\tilde{X}_s,\tilde{Y}_s,X_0,Y_0) + \mathbb{P}_{u_0,v_0}\left(\int_0^t AF(\tilde{X}_s,\tilde{Y}_s,X_r,Y_r)\,dr\right)\right) \\ &= \mathbb{P}_{\tilde{u}_0,\tilde{v}_0}\left(F(\tilde{X}_s,\tilde{Y}_s,X_0,Y_0)\right) + \int_0^t \mathbb{P}_{\tilde{u}_0,\tilde{v}_0}\times\mathbb{P}_{u_0,v_0}\left(AF(\tilde{X}_s,\tilde{Y}_s,X_r,Y_r)\right)\,dr. \end{aligned} \tag{2.4}$$

The application of Fubini's Theorem is justified because

$$\begin{aligned} |AF(\tilde{\phi},\tilde{\psi},x,y)| &\le \exp\{-\langle x,\tilde{\phi}\rangle\}\left[\langle x,|Q\tilde{\phi}|\rangle + \langle y,|Q\tilde{\psi}|\rangle + (\gamma^2/4)\langle x,\tilde{\phi}\rangle^2\right] \\ &\le \langle x,|Q\tilde{\phi}|\rangle + \langle y,|Q\tilde{\psi}|\rangle + c, \end{aligned}$$

and so

$$\begin{aligned} &\sup_{s\le t, r\le t} \mathbb{P}_{\tilde{u}_0,\tilde{v}_0}\times\mathbb{P}_{u_0,v_0}\left(|AF(\tilde{X}_s,\tilde{Y}_s,X_r,Y_r)|\right) \\ &\quad\le \sup_{s\le t,r\le t} 2\sum_j\sum_k \mathbb{P}_{u_0,v_0}\big(X_r(j)\big)|q_{jk}|\mathbb{P}_{\tilde{u}_0,\tilde{v}_0}\big(\tilde{X}_s(k)\big) + c \\ &\quad\le \sup_{s\le t,r\le t} 2\sum_j\sum_k \big(P_r u_0(j) + P_r v_0(j)\big)|q_{jk}|\big(P_s\tilde{u}_0(k) + P_s\tilde{v}_0(k)\big) + c \\ &\quad< \infty \end{aligned}$$

by Theorem 2.2(b)(iii) and a repeated application of (H_2), respectively. Similarly we have from (2.3)

$$f(s,t) = \mathbb{P}_{\tilde{u}_0,\tilde{v}_0}\left(F(\tilde{X}_0,\tilde{Y}_0,X_t,Y_t)\right) + \int_0^s \mathbb{P}_{\tilde{u}_0,\tilde{v}_0}\times\mathbb{P}_{u_0,v_0}\left(AF(\tilde{X}_r,\tilde{Y}_r,X_t,Y_t)\right)\,dr. \tag{2.5}$$

A standard lemma (Lemma 4.4.10 of Ethier and Kurtz (1986)) now gives

$$f(t,0)-f(0,t)=\int_0^t f_1(s,t-s)-f_2(s,t-s)\,ds=0 \quad \text{by (2.4) and (2.5),}$$

where f_1 and f_2 are the derivatives of the absolutely continuous functions $f(\cdot,t)$ and $f(s,\cdot)$, respectively. This proves (b) for the laws of any solutions of $(LS)_{u_0,v_0}$ and $(LS)_{\tilde u_0,\tilde v_0}$.

(a) The above result and the existence of a solution to $(LS)_{\tilde u_0,\tilde v_0}$ for any $\tilde u_0,\tilde v_0\in M_{rap}$ (Theorem 2.2) show that the left side of (2.1) is unique for any solution of $(LS)_{u_0,v_0}$ (which exist by Theorem 2.2) and any $(\tilde X_0,\tilde Y_0)\in\tilde E$. Lemma 2.3(b) and the fact that $(X_t,Y_t)\in E$ a.s. (Theorem 2.2) show that the law of (X_t,Y_t) is unique for any $t>0$, and for any solution of $(LS)_{u_0,v_0}$. In fact we only need the fact that (X,Y) satisfies the martingale problem for A defined above on $D(A)=\{F_{\tilde\phi,\tilde\psi}:(\tilde\phi,\tilde\psi)\in\tilde E\}$. A standard result on martingale problems (e.g. see Theorem 4.4.2 of Ethier-Kurtz (1986) and Theorem 6.2.3 of Stroock-Varadhan (1979)) shows the law of $(X_\cdot,Y_\cdot)$ and hence $(u_\cdot,v_\cdot)$ is unique. Again we use Theorem 2.2 to see that this law is on Ω_{tem}. ∎

Remark 2.5. (a) The above argument shows that $\mathbb{P}_{u_0,v_0}$ is the unique law on Ω_{tem} which solves the martingale problem for A given by (2.1) on $D(A)=\{F_{\tilde\phi,\tilde\psi}:(\tilde\phi,\tilde\psi)\in E_f\}$ and initial conditions $(u_0,v_0)\in(M_{tem})^2$. We will refer to this martingale problem as $(MP)_{u_0,v_0}$.

(b) Theorem 1.1 is contained in Theorems 2.2 and 2.4.

Corollary 2.6. If $u_0,v_0\in M_{tem}$ and $\theta>0$, then

$$\mathbb{P}_{\theta u_0,\theta v_0}(\cdot)=\mathbb{P}_{u_0,v_0}(\theta(u,v)\in\cdot).$$

Proof. $\mathbb{P}_{u_0,v_0}\big(\theta(u,v)\in\cdot\big)$ solves $(MP)_{\theta u_0,\theta v_0}$ (alternatively consider $(LS)_{\theta u_0,\theta v_0}$) and so this follows by the uniqueness in Remark 2.5. ∎

Notation. If $f:(M_{tem})^2\to\mathbb{R}$ is bounded and measurable let $\overline{P}_t f(u,v)=\mathbb{P}_{u,v}\big(f(U_t,V_t)\big)$ for $(u,v)\in(M_{tem})^2$. $C_b(S)$ denotes the space of bounded continuous functions on a metric space S.

The Feller and strong Markov properties of solutions to (LS) are now easy consequences of the duality result in Theorem 2.4.

Corollary 2.7. (a) $\overline{P}_t:C_b\big((M_{tem})^2\big)\to C_b\big((M_{tem})^2\big)\ \ \forall t\ge0$.

(b) Let (u,v,B,W) solve $(LS)_{u_0,v_0}$ on some $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$. If T is an a.s. finite $(\mathcal{F}_t)$-stopping time, then for any bounded measurable f on $(M_{tem})^2$ and any $t\ge0$

$$\mathbb{P}\left(f(u_{T+t},v_{T+t})\mid\mathcal{F}_T\right)=\overline{P}_t f(u_T,v_T)\quad \mathbb{P}\text{-a.s.}$$

Proof. (a) Let $(u_0^n,v_0^n)\to(u_0,v_0)$ in $(M_{tem})^2$ and let P_n (respectively P_∞) be the distribution of $(X_t,Y_t)=(u_t+v_t,u_t-v_t)$ on E under $\mathbb{P}_{u_0^n,v_0^n}$ (respectively, $\mathbb{P}_{u_0,v_0}$). We check the hypotheses of Lemma 2.3(c). Let $\lambda>0$ and choose $\tilde\lambda>0$ so that $\lambda'(\tilde\lambda)<\lambda$ (recall (H_2)). Then

$$\begin{aligned}
\mathbb{P}_{u_0^n,v_0^n}(\langle X_t,\phi_{-\lambda}\rangle) &= \sum_j\sum_k \big(u_0^n(j)+v_0^n(j)\big)p_t(j,k)\phi_{-\lambda}(k) \quad \text{Theorem 2.2(b)(iii))}\\
&\le \sum_j\sum_k \big(u_0^n(j)+v_0^n(j)\big)c_{1.1}(t,\tilde\lambda)\phi_{-\tilde\lambda}(j)\phi_{\lambda'(\tilde\lambda)-\lambda}(k) \quad \text{(by } (H_2) \text{ and}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{symmetry of } p_t)\\
&\le c(t,\tilde\lambda,\lambda)\langle u_0^n+v_0^n,\phi_{-\tilde\lambda}\rangle
\end{aligned}$$

and the latter is bounded uniformly in n because $u_0^n+v_0^n\to u_0+v_0$ in M_{tem}. It is immediate from Theorem 2.4(b) that for $(\phi,\psi)\in E_f$

$$\begin{aligned}
\lim_{n\to\infty} P_n(F_{\phi,\psi}) &= \lim_{n\to\infty}\mathbb{P}_{\frac{\phi+\psi}{2}\,\frac{\phi-\psi}{2}}\Big(\exp\{-\langle u_0^n+v_0^n,\tilde X_t\rangle + i\langle u_0^n-v_0^n,\tilde Y_t\rangle\}\Big)\\
&= \mathbb{P}_{\frac{\phi+\psi}{2}\,\frac{\phi-\psi}{2}}\Big(\exp\{-\langle u_0+v_0,\tilde X_t\rangle + i\langle u_0-v_0,\tilde Y_t\rangle\}\Big) \quad \text{(since } (\tilde X_t,\tilde Y_t)\in\tilde E \text{ a.s.}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by Theorem 2.2(c))}\\
&= P_\infty(F_{\phi,\psi}).
\end{aligned}$$

Lemma 2.3(c) now shows that $P_n \xrightarrow{w} P_\infty$ on E and so $\overline{P}_t f(u_0^n,v_0^n)\to\overline{P}_t f(u_0,v_0)$ for $f\in C_t\big((M_{tem})^2\big)$.

(b) This is a standard consequence of the uniqueness in Theorem 2.4 and Remark 2.5. For example, see Theorem 4.4.2(c) of Ethier and Kurtz (1986). The required measurability is clear from (a) or Theorem 4.4.6 of the same reference. ∎

Remark 2.8. It is easy to use the previous result and standard tightness arguments, to see that $(u_0,v_0)\to\mathbb{P}_{u_0,v_0}$ is a continuous map into the space of probabilities on Ω_{tem} with the topology of weak convergence.

Proof of Theorem 1.4. Let $(\phi,\psi)\in\tilde E$. Theorem 2.2(d) implies that $\langle\tilde u_t,1\rangle$ and $\langle\tilde v_t,1\rangle$ are non-negative continuous martingales under $\mathbb{P}_{\frac{\phi+\psi}{2}\,\frac{\phi-\psi}{2}}\equiv\mathbb{P}'$ and hence converge $\mathbb{P}'$-a.s. as $t\to\infty$ to $\langle\tilde u_\infty,1\rangle$ and $\langle\tilde v_\infty,1\rangle$, respectively (say) by the Martingale Convergence Theorem. Theorem 2.4(b) therefore shows us that

$$\begin{aligned}
\lim_{t\to\infty}\mathbb{P}_{u,v}\big(F_{\phi,\psi}(X_t,Y_t)\big) &= \lim_{t\to\infty}\mathbb{P}'\left(\exp\{-(u+v)\langle\tilde u_t+\tilde v_t,1\rangle + i(u-v)\langle\tilde u_t-\tilde v_t,1\rangle\}\right)\\
&= \mathbb{P}'\left(\exp\{-(u+v)(\langle\tilde u_\infty,1\rangle+\langle\tilde v_\infty,1\rangle)+i(u-v)(\langle\tilde u_\infty,1\rangle-\langle\tilde v_\infty,1\rangle)\}\right). \qquad (2.6)
\end{aligned}$$

Theorem 2.2b(iii) implies that for $\lambda>0$

$$\mathbb{P}_{u,v}(\langle X_t,\phi_{-\lambda}\rangle) = (u+v)\langle\phi_{-\lambda},1\rangle<\infty.$$

Lemma 2.3(c) shows that $\mathbb{P}_{u,v}\big((X_t,Y_t)\in\cdot\big)$ converges weakly on E as $t\to\infty$ and since (u_t,v_t) is a linear function of (X_t,Y_t), therefore $\mathbb{P}_{u,v}\big((u_t,v_t)\in\cdot\big)$ also converges weakly on $(M_{tem})^2$ as $t\to\infty$ to a limit which we denote by $\mathbb{P}_{u,v}\big((u_\infty,v_\infty)\in\cdot\big)$. Lemma 2.3(c) and (2.6) also imply

$$\mathbb{P}_{u,v}\big(F_{\phi,\psi}(u_\infty+v_\infty,u_\infty-v_\infty)\big) = \mathbb{P}'\left(\exp\{-(u+v)(\langle\tilde u_\infty,1\rangle+\langle\tilde v_\infty,1\rangle)+i(u-v)(\langle\tilde u_\infty,1\rangle-\langle\tilde v_\infty,1\rangle)\}\right). \qquad (2.7)$$

Turning next to the mean measures of u_∞ and v_∞, let $\phi \in M_{rap}$ and note that

$$\begin{aligned}
\mathbb{P}_{u,v}(\langle u_\infty + v_\infty, \phi\rangle) &= -\frac{d}{d\lambda}\Big|_{\lambda=0} \mathbb{P}_{u,v}\left(\exp\{-\lambda\langle u_\infty + v_\infty, \phi\rangle\}\right) \\
&= -\frac{d}{d\lambda}\Big|_{\lambda=0} \mathbb{P}_{\lambda\phi/2,\lambda\phi/2}\left(\exp\{-(u+v)(\langle \tilde u_\infty, 1\rangle + \langle \tilde v_\infty, 1\rangle) + i(u-v)(\langle \tilde u_\infty, 1\rangle \right. \\
&\qquad\qquad\qquad\qquad \left. - \langle \tilde v_\infty, 1\rangle)\}\right) \\
&\qquad\qquad\qquad\qquad \text{(by (2.7))} \\
&= -\frac{d}{d\lambda}\Big|_{\lambda=0} \mathbb{P}_{\phi/2,\phi/2}\left(\exp\{-(u+v)\lambda(\langle \tilde u_\infty, 1\rangle + \langle \tilde v_\infty, 1\rangle) + i(u-v)\lambda(\langle \tilde u_\infty, 1\rangle \right. \\
&\qquad\qquad\qquad\qquad \left. - \langle \tilde v_\infty, 1\rangle)\}\right) \\
&\qquad\qquad\qquad\qquad \text{(by Corollary 2.6)} \\
&= \mathbb{P}_{\phi/2,\phi/2}\left((u+v)\big(\langle \tilde u_\infty, 1\rangle + \langle \tilde v_\infty, 1\rangle\big)\right) + i(u-v)\mathbb{P}_{\phi/2,\phi/2}(\langle \tilde u_\infty, 1\rangle - \langle \tilde v_\infty, 1\rangle) \\
&= (u+v)\mathbb{P}_{\phi/2,\phi/2}(\langle \tilde u_\infty, 1\rangle + \langle \tilde v_\infty, 1\rangle), \qquad (2.8)
\end{aligned}$$

as the imaginary part must vanish since the left side is real-valued. By the Dubins-Schwarz theorem and Theorem 2.2(d) under $\mathbb{P}_{\phi/2,\phi/2}$, $(\langle \tilde u_t, 1\rangle, \langle \tilde v_t, 1\rangle)$ is equal in law to $B(A_t)$ where B is a planar Brownian motion starting at $\frac{1}{2}(\langle \phi, 1\rangle, \langle \phi, 1\rangle)$ and

$$A_t = \gamma \int_0^t \langle \tilde u_s, \tilde v_s\rangle \, ds \le T = \inf\{s : B_s^1 B_s^2 = 0\}.$$

Standard estimates show that $P(T > t) \le c(t+1)^{-1}$ and so T^p is integrable for $0 < p < 1$. Burkholders inequality therefore shows that $\sup_t \langle \tilde u_t + \tilde v_t, 1\rangle^{2p}$ is integrable for $0 < p < 1$ and therefore $\{\langle \tilde u_t + \tilde v_t, 1\rangle : t \ge 0\}$ is a uniformly integrable martingale. (2.8) therefore allows us to conclude that

$$\begin{aligned}
\mathbb{P}_{u,v}(\langle u_\infty + v_\infty, \phi\rangle) &= (u+v)\lim_{t\to\infty} \mathbb{P}_{\phi/2,\phi/2}(\langle \tilde u_t, 1\rangle + \langle \tilde v_t, 1\rangle) \\
&= (u+v)\langle \phi, 1\rangle \quad \text{(Theorem 2.2(b)(iii))} \\
&= \lim_{t\to\infty} \mathbb{P}_{u,v}(\langle u_t + v_t, \phi\rangle) \quad \text{(Theorem 2.2(b)(iii) again).}
\end{aligned}$$

This and the weak convergence of $\langle u_t + v_t, \phi\rangle$ to $\langle u_\infty + v_\infty, \phi\rangle$ imply that $\{\langle u_n + v_n, \phi\rangle : n \in \mathbb{N}\}$ is uniformly integrable and so the same is true of $\{u_n(\phi)\}$ and $\{v_n(\phi)\}$. This proves that

$$\mathbb{P}_{u,v}(\langle u_\infty, \phi\rangle) = \lim_{n\to\infty} \mathbb{P}_{u,v}(\langle u_n, \phi\rangle) = u\langle \phi, 1\rangle.$$

Set $\phi = 1_{\{k\}}$ to conclude $\mathbb{P}_{u,v}(u_\infty(k)) = u$ and the same argument gives $\mathbb{P}_{u,v}(v_\infty(k)) = v$.

The fact that $\nu(\cdot) = \mathbb{P}_{u,v}((u_\infty, v_\infty) \in \cdot)$ is a stationary initial distribution for the Markov process (u_t, v_t) is an easy and well-known consequence of the above weak convergence and the Feller property (Corollary 2.7(a)). Finally if (q_{jk}) is spatially homogeneous then uniqueness to the martingale problem (MP) (recall Remark 2.5) shows that (u_t, v_t) is a stationary random field under $\mathbb{P}_{u,v}$ and so the same is true of (u_∞, v_∞). ∎

As it will be useful later we record (2.7) as

Corollary 2.9. The equilibrium distribution for (u_t, v_t) constructed in Theorem 1.4 satisfies

$$\begin{aligned} \mathbb{P}_{u,v}\big(F_{\phi,\psi}(u_\infty + v_\infty, u_\infty - v_\infty)\big) = \mathbb{P}_{(\phi+\psi)/2,(\phi-\psi)/2}\,&(\exp\{-(u+v)(\langle \tilde{u}_\infty, 1\rangle + \langle \tilde{v}_\infty, 1\rangle) \\ &+ i(u-v)(\langle \tilde{u}_\infty, 1\rangle - \langle \tilde{v}_\infty, 1\rangle)\}) \\ &\forall(\phi,\psi) \in \tilde{E}. \end{aligned} \tag{2.9}$$

Remark. Of course (2.9) in fact characterizes $\mathbb{P}_{u,v}\big((u_\infty, v_\infty) \in \cdot\big)$ (by Lemma 2.3(b)). Hence it codes up properties of this equilibrium distribution in terms of the laws of $(\langle \tilde{u}_\infty, 1\rangle, \langle \tilde{v}_\infty, 1\rangle)$ for rapidly decreasing initial conditions. In the next section we will study properties of these laws which will allow us to decide when co-existence of types is possible for these equilibrium laws (in Section 4). The fact that the self-dual relation (2.9) is nonetheless very useful through its linking of finite and infinite initial conditions, is reminiscent of the situation for infinite linear systems in Chapter IX of Liggett (1985).

3. Co-existence for Integrable Initial Conditions.

In this section we prove Theorem 1.2. Recall from Theorem 2.2(d) that for $u_0, v_0 \in M_F$, under $\mathbb{P}_{u_0,v_0}$ $\langle u_t, 1\rangle$ and $\langle v_t, 1\rangle$ are orthogonal continuous L^2 martingales with common square function

$$\gamma A_t \equiv \gamma \int_0^t \langle u_s, v_s\rangle\, ds.$$

Recall also that $\langle u_\infty, 1\rangle$ and $\langle v_\infty, 1\rangle$ denote their respective a.s. limits as $t \to \infty$.

Proof of Theorem 1.2 (a). The mean value results from Theorem 2.2(b)(iii) show that

$$\begin{aligned} \mathbb{P}_{u_0,v_0}(A_\infty) &= \int_0^\infty \sum_i \left(\sum_j u_0(j) p_s(j,i)\right)\left(\sum_k v_0(k) p_s(k,i)\right) ds \\ &= \sum_j \sum_k u_0(j) v_0(k) \int_0^\infty p_{2s}(j,k)\, ds \\ &\le \langle u_0, 1\rangle \langle v_0, 1\rangle \sup_{j,k} \frac{1}{2} g_\infty(j,k). \end{aligned}$$

The strong Markov property of ξ therefore shows that

$$\mathbb{P}_{u_0,v_0}(A_\infty) \le \langle u_0, 1\rangle \langle v_0, 1\rangle \sup_k \frac{1}{2} g_\infty(k,k) < \infty \quad \text{(by hypothesis)}.$$

Doobs strong L^2 inequality for martingales implies

$$\begin{aligned} \mathbb{P}_{u_0,v_0}\left(\sup_t \langle u_t, 1\rangle \langle v_t, 1\rangle\right) \le \mathbb{P}_{u_0,v_0}\left(\sup_t \langle u_t, 1\rangle^2 + \langle v_t, 1\rangle^2\right) &\le c\big(\langle u_0, 1\rangle^2 + \langle v_0, 1\rangle^2 + \mathbb{P}_{u_0,v_0}(A_\infty)\big) \\ &< \infty. \end{aligned}$$

Therefore $\langle u_t, 1\rangle\langle v_t, 1\rangle$ is an H^1-martingale and in particular

$$\mathbb{P}_{u_0,v_0}(\langle u_\infty, 1\rangle\langle v_\infty, 1\rangle) = \langle u_0, 1\rangle\langle v_0, 1\rangle. \tag{3.1}$$

This shows that co-existence of types is possible. ∎

In the proof of Theorem 1.2(b) we work directly with a solution (u, v, B, W) of $(LS)_{u_0,v_0}$ defined on some $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, with $\langle u_0, 1\rangle\langle v_0, 1\rangle > 0$.

Proof of Theorem 1.2(b). To simplify the notation set $\gamma = 1$. Define $p_r(k, j) = 0$ for $r < 0$,

$$M_t^u(j) = \int_0^t \big(u_r(j)v_r(j)\big)^{1/2} dB_r^j, \quad N_s^u(t, j) = \sum_k \int_0^s p_{t-r}(k, j)\, dM_r^u(k),$$

and similarly define $M_t^v(j)$ and $N_s^v(t, j)$. Theorem 2.2(b) shows the above series in k converges uniformly in $s \le t$ in L^2 and that

$$u_t(j) = P_t u_0(j) + N_t^u(t, j), \quad v_t(j) = P_t v_0(j) + N_t^v(t, j) \quad \text{a.s.} \tag{3.2}$$

Some care is needed as $N_\cdot^u(t, j)$ is only defined up to a null set for each $t \ge 0$, and we will need some appropriate versions in what follows. Let $t_n(t) = ([2^n t] + 1)2^{-n}$. Then for t fixed, an L^2 calculation making repeated use of (H_2) shows that

$$\mathbb{P}\left(\sup_{s \le t_n(t)} \left|\sum_{k\in S_n} \int_0^s p_{t_n(t)-r}(k, j)\, dM_r^u(k) - \sum_k \int_0^s p_{t-r}(k, j)\, dM_r^u(k)\right|^2\right) \le c_1 e^{-c_2 n}$$

and so by Borel-Cantelli a (predictable $\times$ Borel) version of $(s, \omega, t) \to N_s^u(t, j)$ (agreeing with $N_\cdot^u(t, j)$ a.s. for each t) is given by

$$N_s^u(t, j) = \begin{cases} \lim\limits_{n\to\infty} \sum\limits_{k\in S_n} \int\limits_0^s p_{t_n(t)-r}(k, j)\, dM_r^u(k) & \text{if the limit exists} \\ 0 & \text{otherwise} \end{cases}.$$

Moreover $s \to N_s^u(t, j)$ is a continuous square integrable martingale for each (t, j). This procedure also implicitly defines (predictable $\times$ Borel) versions of

$$(s, \omega, t) \to \int_0^s p_{t-r}(k, j)\, dM_r^u(k) \quad \text{for each } j, k \quad \text{(and similarly when } u \text{ is replaced by } v).$$

We have from (3.2), for a fixed $T > 0$,

$$\begin{aligned} A_T &= \int_0^T \sum_j \big(P_t u_0(j) + N_t^u(t, j)\big)\big(P_t v_0(j) + N_t^v(t, j)\big)\, dt \\ &= \int_0^T \langle P_t u_0, P_t v_0\rangle\, dt + \int_0^T \langle P_t u_0, N_t^v(t)\rangle\, dt + \int_0^T \langle P_t v_0, N_t^u(t)\rangle\, dt + \int_0^T \langle N_t^u(t), N_t^v(t)\rangle\, dt \\ &\equiv A_T^{(1)} + A_T^{(2)} + A_T^{(3)} + A_T^{(4)}. \end{aligned}$$

One can readily check (using Theorem 2.2) that the series defining the integrands of $A^{(2)}$ and $A^{(3)}$ converge in L^2 uniformly in $t \le T$ and the integrand of $A^{(4)}$ converges in L^1 uniformly in $t \le T$. Considering $A_T^{(2)}$, we have (the limits being in L^2)

$$\begin{aligned} A_T^{(2)} &= \lim_{m\to\infty} \lim_{n\to\infty} \int_0^T \sum_{j\in S_m} \sum_{k\in S_n} P_t u_0(j) \int_0^t p_{t-r}(k,j)\, dM_r^v(k)\, dt \\ &= \lim_{m\to\infty} \lim_{n\to\infty} \sum_{k\in S_n} \int_0^T \left(\int_r^T \sum_{j\in S_m} P_t u_0(j) p_{t-r}(k,j)\, dt \right) dM_r^v(k). \end{aligned}$$

In the last line a stochastic Fubini argument has been used (see p. 116 of Ikeda and Watanabe (1981) and recall we have chosen a jointly measurable version of $(t,\omega) \to \int_0^t p_{t-r}(k,j)\, dM_r^v(k)$). It is easy to see that the above L^2 limit exists jointly in (m,n). Therefore we may interchange limits and take the limit in m through the integrals to conclude

$$A_T^{(2)} = \sum_k \int_0^T \int_r^T P_{2t-r} u_0(k)\, dt\, dM_r^v(k) = \sum_k \int_0^T h^{u_0}(r,k)\, dM_r^v(k), \tag{3.3}$$

where $h^{u_0}(r,k) = \frac{1}{2} \int_r^{2T-r} P_s u_0(k)\, ds$. Similarly we have

$$A_T^{(3)} = \sum_k \int_0^T h^{v_0}(r,k)\, dM_r^u(k).$$

The above series converge in L^2. Define $A_t^{(2)} = \sum_k \int_0^t h^{u_0}(r,k)\, dM_r^v(k)$ and similarly define $A_t^{(3)}$ for $t \le T$. Integrating by parts in the expression for $A_T^{(4)}$ we have (the limits now being in L^1)

$$\begin{aligned} A_T^{(4)} &= \lim_{m\to\infty} \int_0^T \left[\sum_{j\in S_m} \int_0^t N_r^u(t,j)\, dN_r^v(t,j) + \int_0^t N_r^v(t,j)\, dN_r^u(t,j) \right] dt \\ &= \lim_{m\to\infty} \lim_{n\to\infty} \left[\int_0^T \sum_{j\in S_m} \sum_{k\in S_n} \int_0^t N_r^u(t,j) p_{t-r}(k,j)\, dM_r^v(k)\, dt \right. \\ &\quad \left. + \int_0^T \sum_{j\in S_m} \sum_{k\in S_n} \int_0^t N_r^v(t,j) p_{t-r}(k,j)\, dM_r^u(k)\, dt \right] \end{aligned}$$

$$= \lim_{m\to\infty} \lim_{n\to\infty} \Bigg[\sum_{k\in S_n} \int_0^T \sum_{j\in S_m} \int_r^T N_r^u(t,j) p_{t-r}(k,j)\,dt\,dM_r^v(k)$$
$$+ \sum_{k\in S_n} \int_0^T \sum_{j\in S_m} \int_r^T N_r^v(t,j) p_{t-r}(k,j)\,dt\,dM_r^u(k) \Bigg].$$

In the last line we again have used a stochastic Fubini theorem (the boundedness condition on p. 116 of Ikeda and Watanabe (1981) may be weakened to an integrability condition and we have already addressed the measurability requirements). The L^1-convergence in the above (or in the original sums defining $A_T^{(4)}$) is joint in (m,n) and we may therefore interchange limits and take the limit in m through the stochastic integral. If

$$g^u(r,k) = \sum_j \int_r^T N_r^u(t,j) p_{t-r}(k,j)\,dt$$

and g^v is defined similarly, this gives

$$A_T^{(4)} = \sum_k \int_0^T g^u(r,k)\,dM_r^v(k) + \sum_k \int_0^T g^v(r,k)\,dM_r^u(k) \equiv A_T^{(4.1)} + A_T^{(4.2)}.$$

The sums converge in L^1. Define $A_t^{(4)}$, $A_t^{(4.1)}$ and $A_t^{(4.2)}$ as above but with $t \le T$ (T is fixed in the definitions of g^u and g^v, however).

Our plan centers on the fact that A_T converges as $T \to \infty$ as it is the square function of a non-negative (and hence convergent) martingale $\langle u_T, 1\rangle$. We will show that as A_T levels off, so do $A_T^{(i)}$ for $i = 2,3,4$ but that if $\langle u_T,1\rangle\langle v_T,1\rangle$ remains bounded away from zero then $A_T^{(1)}$ will continue to grow. This forces A_T to continue to grow, a contradiction. Our first step is to control $A_T^{(i)}$ ($i \ge 2$) in terms of A_T and for this we will bound the square functions $\langle A^{(i)}\rangle_t$ of the martingales $A_t^{(i)}$, $t \le T$ (recall T is fixed).

Consider first $A^{(4.1)}$. If $r \le t(\le T)$ is fixed then (3.2) shows that w.p.1

$$\begin{aligned} |N_r^u(t,j)| &\le \mathbb{P}(u_t(j) \mid \mathcal{F}_r) + P_t u_0(j) \\ &= \mathbb{P}_{u_r,v_r}\big(u_{t-r}(j)\big) + P_t u_0(j) \quad \text{(Markov property – see Corollary 2.7)} \\ &= P_{t-r}u_r(j) + P_t u_0(j) \quad \text{(Theorem 2.2(b)(iii))}. \end{aligned}$$

This shows that for $r(\le T)$ fixed, w.p.1

$$\begin{aligned} |g^u(r,k)| &\le \int_r^T P_{2(t-r)}u_r(k) + P_{2t-r}u_0(k)\,dt \\ &\le \frac{1}{2}\int_0^{2T} \sum_j p_s(k,j)\big(u_r(j) + u_0(j)\big)\,ds \\ &\le \big((\langle u_r,1\rangle + \langle u_0,1\rangle)/2\big) \sup_j g_{2T}(j,k). \end{aligned}$$

If $\langle u_T^*, 1\rangle = \sup_{r\le T}\langle u_r, 1\rangle$ and $\|g_{2T}\|_\infty = \sup_j g_{2T}(j,j) (= \sup_{j,k} g_{2T}(j,k)$ by the strong Markov property of ξ), then this shows that

$$\begin{aligned}&\langle A^{(4.1)}\rangle_T \le \langle u_T^*, 1\rangle^2 \|g_{2T}\|_\infty^2 A_T, \quad \text{and analogously}\\ &\langle A^{(4.2)}\rangle_T \le \langle v_T^*, 1\rangle^2 \|g_{2T}\|_\infty^2 A_T.\end{aligned} \tag{3.4}$$

Turning next to $A^{(2)}$, and recalling the definition of h^{u_0} in (3.3), we have w.p.1

$$|h^{u_0}(r,k)| \le \frac{1}{2}\int_0^{2T} \sum_j p_s(k,j)u_0(j)\,ds \le \frac{1}{2}\langle u_0, 1\rangle \|g_{2T}\|_\infty$$

and therefore

$$\begin{aligned}&\langle A^{(2)}\rangle_T \le \langle u_0, 1\rangle^2 \|g_{2T}\|_\infty^2 A_T, \quad \text{and analogously}\\ &\langle A^{(3)}\rangle_T \le \langle v_0, 1\rangle^2 \|g_{2T}\|_\infty^2 A_T.\end{aligned} \tag{3.5}$$

Let N_t denote the martingale $\sum_2^4 A_t^{(i)}$, $t \le T$. The Kunita-Watanabe inequalities, (3.4), and (3.5) show that for some universal $c_1 > 0$,

$$\langle N\rangle_T \le c_1\big(\langle u_T^*, 1\rangle^2 + \langle v_T^*, 1\rangle^2\big)\|g_{2T}\|_\infty^2 A_T. \tag{3.6}$$

Next we establish a lower bound on $A_T^{(1)}$. First choose $m = m(u_0, v_0)$ such that

$$\sum_{|k|\le m} u_0(k) \ge \frac{1}{2}\langle u_0, 1\rangle \quad \text{and} \quad \sum_{|k|\le m} v_0(k) \ge \frac{1}{2}\langle v_0, 1\rangle.$$

Our recurrence hypothesis on ξ shows that we may then choose $T_1 = T_1(u_0, v_0) < \infty$, a measurable function of (u_0, v_0), such that

$$P_j(\xi_t = k \quad \text{for some} \quad t \le T_1) \ge \frac{1}{2} \quad \forall |j| \le m, |k| \le m, \quad \text{and} \tag{3.7}$$

$$\frac{c_{1.2}}{32}\langle u_0, 1\rangle\langle v_0, 1\rangle \|g_{2T_1}\|_\infty \ge 2 \quad \text{and} \quad T_1 > \sup_{|j|\le m} T_{1.2}(j) \tag{3.8}$$

In (3.8) $c_{1.2}$ and $T_{1.2}$ are as in (1.1) and we have used the fact that $\lim_{T\to\infty}\|g_{2T}\|_\infty = \infty$ by the recurrence hypothesis and $\sup_i |q_{ii}| \le \infty$. We are also using the fact that $\langle u_0, 1\rangle\langle u_0, 1\rangle > 0$. To complete the definition of T_1, set $T_1(u_0, v_0) = \infty$ if $\langle u_0, 1\rangle\langle v_0, 1\rangle = 0$. If $T \ge T_1(u_0, v_0)$, then

$$\begin{aligned}A_T^{(1)} &= \sum_j\sum_k u_0(j)v_0(k)\frac{1}{2}g_{2T}(j,k)\\ &\ge \frac{1}{2}\sum_{|j|\le m}\sum_{|k|\le m} u_0(j)v_0(k)\inf\{g_{2T}(j,k) : |j| \le m, |k| \le m\}\\ &\ge \frac{1}{8}\langle u_0, 1\rangle\langle v_0, 1\rangle \inf\{P_j(\xi_t \,\text{hits}\; k \;\text{before}\; T)g_T(k,k) : |j| \le m, |k| \le m\}\\ &\ge \frac{c_{1.2}}{16}\langle u_0, 1\rangle\langle v_0, 1\rangle \|g_T\|_\infty \quad \text{(by (1.1) and (3.7))}.\end{aligned}$$

The strong Markov property implies that $g_{2T}(k,k) \le 2g_T(k,k)$ and so

$$A_T^{(1)} \ge \frac{c_{1.2}}{32}\langle u_0,1\rangle\langle v_0,1\rangle\|g_{2T}\|_\infty \quad \text{for} \quad T \ge T_1(u_0,v_0). \tag{3.9}$$

Set $c_2 = c_{1.2}/32$ and choose $R > 0$ so that

$$R \ge \max\bigl(2(\langle u_0,1\rangle + \langle v_0,1\rangle), \langle u_0,1\rangle\langle v_0,1\rangle\bigr). \tag{3.10}$$

Let B_t denote a real-valued Brownian motion starting at x under P_x. If $T \ge T_1(u_0,v_0)$, then

$$\begin{aligned}\mathbb{P}(A_T \le 1) &\le \mathbb{P}\left(c_2\langle u_0,1\rangle\langle v_0,1\rangle\|g_{2T}\|_\infty + N_T \le 1, A_T \le 1\right) \quad \text{(by (3.9))}\\ &\le \mathbb{P}\left(N_T \le -c_2\langle u_0,1\rangle\langle v_0,1\rangle\|g_{2T}\|_\infty/2, A_T \le 1, (\langle u_T^*,1\rangle \vee \langle v_T^*,1\rangle) \le R\right)\\ &+ \mathbb{P}\left(\langle u_T^*,1\rangle > R, A_T \le 1\right) + \mathbb{P}\left(\langle v_T^*,1\rangle > R, A_T \le 1\right) \quad \text{(by (3.8))}\\ &\le \mathbb{P}\left(N_T \le -c_2\langle u_0,1\rangle\langle v_0,1\rangle\|g_{2T}\|_\infty/2, \langle N\rangle_T \le 2c_1R^2\|g_{2T}\|_\infty^2\right)\\ &+ P_{\langle u_0,1\rangle}\left(\sup_{s\le 1} B_s > R\right) + P_{\langle v_0,1\rangle}\left(\sup_{s\le 1} B_s > R\right) \quad \text{(by (3.6)).}\end{aligned} \tag{3.11}$$

The reflection principle, (3.10), and a standard Gaussian tail estimate bounds the sum of the last two terms by

$$2P_0\left(\sup_{s\le 1} B_s > R/2\right) = 4P_0\left(B_1 > R/2\right) \le \frac{8}{R}e^{-R^2/8}. \tag{3.12}$$

The first term on the right side of (3.11) is dominated by

$$\begin{aligned}&P_0\bigl(\inf\{B_t : t \le 1\} \le -c_2\langle u_0,1\rangle\langle v_0,1\rangle R^{-1}(2c_1)^{-1/2}/2\bigr)\\ &\qquad = 1 - P_0\left(|B_1| \le c_2(2c_1)^{-1/2}2^{-1}\langle u_0,1\rangle\langle v_0,1\rangle R^{-1}\right) \text{ (reflection principle)}\\ &\qquad \le 1 - c_3\langle u_0,1\rangle\langle v_0,1\rangle R^{-1},\end{aligned} \tag{3.13}$$

where in the last line we use $R \ge \langle u_0,1\rangle\langle v_0,1\rangle$ by (3.10) and $c_3 > 0$ is universal. Now set

$$R = R(\langle u_0,1\rangle, \langle v_0,1\rangle) = \max\left(2(\langle u_0,1\rangle + \langle v_0,1\rangle), \langle u_0,1\rangle\langle v_0,1\rangle, (8|\ln(16/c_3\langle u_0,1\rangle\langle v_0,1\rangle)|)^{1/2}\right).$$

Use (3.12) and (3.13) in (3.11) and conclude that for $T \ge T_1(u_0,v_0)$,

$$\begin{aligned}\mathbb{P}(A_T > 1) &\ge c_3\langle u_0,1\rangle\langle v_0,1\rangle R^{-1} - 8R^{-1}\exp(-R^2/8)\\ &\ge c_3\langle u_0,1\rangle\langle v_0,1\rangle\bigl(2R(\langle u_0,1\rangle, \langle v_0,1\rangle)\bigr)^{-1}\\ &\equiv q(\langle u_0,1\rangle, \langle v_0,1\rangle).\end{aligned} \tag{3.14}$$

Also define $q(u,0) = q(0,v) = 0$, so that the above remains valid if $\langle v_0,1\rangle\langle v_0,1\rangle = 0$. Note that

$$\inf\{q(u,v) : \delta \le u, v\} = \epsilon(\delta) > 0 \quad \forall \delta > 0. \tag{3.15}$$

Inductively define $T_{n+1} = T_1(u_{T_n}, v_{T_n}) + T_n$, if $T_n < \infty$, and set $T_{n+1} = \infty$ if $T_n = \infty$. Clearly T_n is an $(\mathcal{F}_t)$-stopping time and so by the strong Markov property (Corollary 2.7)

$$\begin{aligned}\mathbb{P}\left(A(T_{n+1}) - A(T_n) \geq 1 \mid \mathcal{F}_{T_n}\right) &= \mathbb{P}_{u_{T_n}, v_{T_n}}\big(A\big(T_1(u_0, v_0)\big) \geq 1\big)\, 1(T_n < \infty) \\ &\geq q(\langle u_{T_n}, 1\rangle, \langle v_{T_n}, 1\rangle) 1(T_n < \infty).\end{aligned} \tag{3.16}$$

By Lévy's conditional version of the Borel-Cantelli Lemma (see 12.15 of Williams (1991)) we have

$$\{A(T_{n+1}) - A(T_n) \geq 1 \ \ i.o.\} \supset \left\{\sum_{n=1}^{\infty} q(\langle u_{T_n}, 1\rangle, \langle v_{T_n}, 1\rangle) 1(T_n < \infty) = \infty\right\} \quad \text{a.s.} \tag{3.17}$$

Since $A(\infty) = \lim_{t\to\infty} A(t)$ is a.s. finite (A is the square function of the nonnegative, and therefore convergent, martingale $\langle u_t, 1\rangle$), the left side of (3.17) is a null set and so

$$\sum_{n=1}^{\infty} q(\langle u_{T_n}, 1\rangle, \langle v_{T_n}, 1\rangle) 1(T_n < \infty) < \infty \quad \text{a.s.} \tag{3.18}$$

If $T_n < \infty$ for all n, then (3.18) and (3.15) imply that (omitting a null set) $\underline{\lim}_{n\to\infty} \langle u_{T_n}, 1\rangle \langle v_{T_n}, 1\rangle = 0$ and hence $\lim_{t\to\infty} \langle u_t, 1\rangle\langle v_t, 1\rangle = 0$ by martingale convergence. If $T_n = \infty$ and n is minimal, then $T_{n-1} < \infty$ and $\langle u_{T_{n-1}}, 1\rangle\langle v_{T_{n-1}}, 1\rangle = 0$. This implies $\langle u_t, 1\rangle\langle v_t, 1\rangle = 0 \quad \forall t \geq T_{n-1}$. In either case we have shown $\langle u_\infty, 1\rangle\langle v_\infty, 1\rangle = 0$ a.s. and so co-existence of types is impossible. ∎

4. The Equilibrium Distributions.

We now use the results of the previous Section to study the equilibrium laws found in Theorem 1.4, which we continue to denote by $\mathbb{P}_{u,v}\big((u_\infty, v_\infty) \in \cdot\big)$ for $u, v \geq 0$. For ease of reference we recall from Corollary 2.9 that

$$\begin{aligned}\mathbb{P}_{u,v}\big(F_{\phi,\psi}(u_\infty + v_\infty, u_\infty - v_\infty)\big) = \mathbb{P}_{(\phi+\psi)/2,(\phi-\psi)/2}\,(\exp\{&-(u+v)(\langle \tilde{u}_\infty, 1\rangle + \langle \tilde{v}_\infty, 1\rangle) \\ &+ i(u-v)(\langle \tilde{u}_\infty, 1\rangle - \langle \tilde{v}_\infty, 1\rangle)\}) \\ \forall (\phi, \psi) \in \tilde{E}.&\end{aligned} \tag{2.9}$$

Proof of Theorem 1.5. Choose $(\phi, \psi) \in \tilde{E}$, let $(\tilde{u}_0, \tilde{v}_0) = \big((\phi+\psi)/2, (\phi-\psi)/2\big)$ $\big(\in (M_{rap})^2\big)$ and let $L(u, v)$ denote the expressions in (2.9). If $h_{b,c}(u, v) = \exp\{-(u+v)b + i(u-v)c\}$ $(b, c \in \mathbb{R})$ and the Laplacian, Δ, is applied separately to the real and imaginary parts of $h_{b,c}$ then one readily checks that $\Delta h_{b,c} = 0$ if $|b| = |c|$. Theorem 1.2(b) implies that $|\langle \tilde{u}_\infty, 1\rangle + \langle \tilde{v}_\infty, 1\rangle| = |\langle \tilde{u}_\infty, 1\rangle - \langle \tilde{v}_\infty, 1\rangle|$ $\mathbb{P}_{\tilde{u}_0,\tilde{v}_0}$-a.s. and so we can differentiate through the expected value on the right side of (2.9) and conclude that $\Delta L(u, v) = 0$ for $u, v > 0$. The right side of (2.9) also shows that L and its first order partial derivatives are bounded and continuous on $[0, \infty)^2$. Here we have used Fatou's lemma to see that $\mathbb{P}_{\tilde{u}_0,\tilde{v}_0}\big(\langle \tilde{u}_\infty, 1\rangle + \langle \tilde{v}_\infty, 1\rangle\big) \leq \langle \tilde{u}_0 + \tilde{v}_0, 1\rangle < \infty$. The left side of (2.9) shows that

$$L(u, 0) = \exp\{-u\langle \phi, 1\rangle + iu\langle \psi, 1\rangle\}, L(0, v) = \exp\{-v\langle \phi, 1\rangle - iv\langle \psi, 1\rangle\}. \tag{4.1}$$

A simple application of Itô's lemma now shows that if $(B^1_\cdot, B^2_\cdot)$, $P^0_{u,v}$, and T are as in the statement of the Theorem, then

$$\begin{aligned} L(u,v) &= \lim_{t\to\infty} P^0_{u,v}\left(L\big(B(T\wedge t)\big)\right) \\ &= P^0_{u,v}\left(L\big(B(T)\big)\right) \\ &= P^0_{u,v}\left(\exp\left\{-\left(B^1_T + B^2_T\right)\langle\phi,1\rangle + i\left(B^1_T - B^2_T\right)\langle\psi,1\rangle\right\}\right) \quad \text{(by (4.1))}. \end{aligned}$$

Returning to (2.9), we have shown that (recall the notation $\overline{B^i_T}$ in the statement of the Theorem)

$$\mathbb{P}_{u,v}\left(F_{\phi,\psi}(u_\infty + v_\infty, u_\infty - v_\infty)\right) = P^0_{u,v}\left(F_{\phi,\psi}\left(\overline{B^1_T} + \overline{B^2_T}, \overline{B^1_T} - \overline{B^2_T}\right)\right) \quad \forall(\phi,\psi)\in\tilde{E}.$$

This and Lemma 2.3(b) imply the required result. ∎

Notation. Define an equivalence relation $\sim$ on $\mathbb{Z}^d$ by $j\sim k$ iff $p_t(j,k)>0$ for some (or equivalently all) $t>0$. Let $[j]=\{k: j\sim k\}$.

Lemma 4.1. Let μ be a probability on $(M_{tem})^2$. For Lebesgue-a.a. $t>0$

$$v_t(j)>0 \text{ implies } v_t(k)>0 \quad \forall k\in[j], \text{ and } u_t(j)>0 \text{ implies } u_t(k)>0 \quad \forall k\in[j] \quad \mathbb{P}_\mu\text{-a.s.}$$

Proof. We may assume $\mu=\delta_{u_0,v_0}$ for $(u_0,v_0)\in(M_{tem})^2$. If L^0_t is the local time at 0 of the semimartingale $v_t(k)$ under $\mathbb{P}_\mu$ (recall Theorem 2.2(b)(iv)), then by Corollary 2 of Yor (1978)

$$L^0_t = \lim_{\epsilon\downarrow 0}\epsilon^{-1}\gamma\int_0^t 1(0<v_s(k)\le\epsilon)u_s(k)v_s(k)\,ds \le \lim_{\epsilon\downarrow 0}\gamma\int_0^t 1(0<v_s(k)\le\epsilon)u_s(k)\,ds = 0.$$

Theorem 2(iv) of Yor (1978) now implies (again using Theorem 2.2(b)(iv) with $\phi=1_{\{k\}}$)

$$0 = L^0_t - L^{0-}_t = 2\int_0^t 1\big(v_s(k)=0\big)\sum_{j\ne k} v_s(j)q_{jk}\,ds \quad \forall t>0 \quad \text{a.s.}$$

This implies $\mathbb{P}_\mu$-a.s. for Lebesgue-a.a. $s\ge 0$, $v_s(j)>0$ implies $v_s(k)>0$ whenever $q_{jk}>0$. By Fubini's theorem we may fix $t\ge 0$ outside a Lebesgue null set and then ω outside a $\mathbb{P}_\mu$-null set so that

$$v_t(j)>0 \quad \text{implies} \quad v_t(k)>0 \quad \text{whenever} \quad j,k\in\mathbb{Z}^d \quad \text{satisfy} \quad q_{jk}>0. \tag{4.2}$$

If $v_t(j)>0$ and $k\in[j]$, there is a finite number of distinct points $j=j_0, j_1,\dots,j_n=k$ such that $\prod_{i=1}^n q_{j_{i-1}j_i}>0$ and so a repeated application of (4.2) implies $v_t(k)>0$. The same argument applies to u. ∎

Proof of Theorem 1.6. (a) We claim it suffices to show

$$\langle v_\infty, 1_{[0]}\rangle > 0 \quad \mathbb{P}_{u,v}\text{-a.s.} \tag{4.3}$$

Fix $t > 0$ outside a Lebesgue null set so that the conclusion of Lemma 4.1 holds with $\mu = \mathbb{P}_{u,v}\big((u_\infty, v_\infty) \in \cdot\big)$. The fact that μ is a stationary distribution for (u,v) (Theorem 1.4) implies that

$$\begin{aligned}
0 &= \mathbb{P}_\mu\left(\langle v_t, 1_{[0]}\rangle > 0, v_t(k) = 0 \quad \text{for some } k \text{ in } [0]\right) \quad (\text{by the choice of } t) \\
&= \mathbb{P}_{u,v}\left(\langle v_\infty, 1_{[0]}\rangle > 0, v_\infty(k) = 0 \quad \text{for some } k \text{ in } [0]\right) \\
&= \mathbb{P}_{u,v}\left(v_\infty(k) = 0 \quad \text{for some } k \text{ in } [0]\right) \quad (\text{by } (4.3)).
\end{aligned}$$

The same reasoning applies to every other equivalence class and to u_∞, and so (a) would follow from (4.3).

Turning to (4.3), we may use Theorem 2.2(b)(ii) with $\phi = 1_{\{0\}}$, $u_0 \equiv u$ and $v_0 \equiv v$ to see that for $t > t_0 > 0$

(4.4)

$$\begin{aligned}
u_t(0) &= u + \sum_k \int_0^{t-t_0} p_{t-s}(k,0)\big(\gamma u_s(k)v_s(k)\big)^{1/2} dB_s^k + \sum_{k\in[0]} \int_{t-t_0}^{t} p_{t-s}(k,0)\left(\gamma u_s(k)v_s(k)\right)^{1/2} dB_s^k \\
&= u + M_1(t-t_0) + M_2(t).
\end{aligned}$$

Let $\epsilon > 0$. Theorem 2.2(b) shows that

$$\begin{aligned}
\mathbb{P}_{u,v}\left(\langle M_1\rangle_{t-t_0}\right) &= \gamma uv \sum_k \int_0^{t-t_0} p_{t-s}(k,0)^2\, ds \\
&\le \gamma uv \int_{t_0}^{\infty} p_{2r}(0,0)\, dr \qquad (4.5) \\
&< \epsilon^4,
\end{aligned}$$

where the last line is valid providing we choose $t_0 = t_0(\epsilon)$ sufficiently large (recall that $g_\infty(0,0) < \infty$). Having fixed such a t_0, note that if $k \in [0]$, then (H_1) and (H_2) imply

$$\sup_{r\le t_0} p_r(j,k) \le c_{1.1}(t_0,1)1_{[0]}(j)e^{\lambda'(1)|k|-|j|}. \tag{4.6}$$

Let $\tilde{\phi}_{-1}(k) = 1_{[0]}(k)\phi_{-1}(k)$ and $(\mathcal{F}_t)$ denote the canonical right continuous filtration on Ω_{tem}. Then for $t > t_0$,

$$\begin{aligned}
&\mathbb{P}_{u,v}\left(\langle M_2\rangle_t > \epsilon^3 \mid \mathcal{F}_{t-t_0}\right) \\
&\quad = \mathbb{P}_{u_{t-t_0}, v_{t-t_0}}\left(\sum_{k\in[0]} \int_0^{t_0} p_{t_0-r}(k,0)^2 \gamma u_r(k) v_r(k)\, dr > \epsilon^3\right) \\
&\quad \le \epsilon^{-3} \int_0^{t_0} \sum_{k\in[0]} p_{t_0-r}(k,0)^2 \gamma \langle u_{t-t_0}, p_r(\cdot,k)\rangle\langle v_{t-t_0}, p_r(\cdot,k)\rangle\, dr \quad (\text{Theorem 2.2(b)(iii)}) \qquad (4.7) \\
&\quad \le \epsilon^{-3} c_{1.1}(t_0,1)^2 \gamma \int_0^{t_0} \sum_{k\in[0]} p_{t_0-r}(k,0) e^{2\lambda'(1)|k|}\, dr \langle u_{t-t_0}, \tilde{\phi}_{-1}\rangle\langle v_{t-t_0}, \tilde{\phi}_{-1}\rangle \quad (\text{by } (4.6)) \\
&\quad \le \epsilon^{-3} c(t_0)\langle u_{t-t_0}, \tilde{\phi}_{-1}\rangle\langle v_{t-t_0}, \tilde{\phi}_{-1}\rangle \quad (\text{by } (H_2)).
\end{aligned}$$

We also have

$$
\begin{aligned}
\mathbb{P}_{u,v}\left(v_t(0) > \epsilon \mid \mathcal{F}_{t-t_0}\right) &= \mathbb{P}_{u_{t-t_0},v_{t-t_0}}(v_{t_0}(0) > \epsilon) \\
&\le \epsilon^{-1}\langle v_{t-t_0}, p_{t_0}(\cdot,0)\rangle \\
&\le \epsilon^{-1} c_{1,1}(t_0,1)\langle v_{t-t_0}, \tilde{\phi}_{-1}\rangle \quad \text{(by (4.6))}.
\end{aligned} \tag{4.8}
$$

Let $A_\delta(t) = \{\omega : \langle u_{t-t_0}, \tilde{\phi}_{-1}\rangle\langle v_{t-t_0}, \tilde{\phi}_{-1}\rangle < \delta, \langle v_{t-t_0}\tilde{\phi}_{-1}\rangle < \delta\}$. (4.7) and (4.8) show that for $t > t_0(\epsilon)$ and $\delta < \delta_0(\epsilon)$ (for some $\delta_0(\epsilon) > 0$ independent of the choice of $t > t_0(\epsilon)$), on $A_\delta(t)$ we have

$$
\begin{aligned}
\mathbb{P}_{u,v}\left(|M_2(t)| > \epsilon \quad \text{or} \quad v_t(0) > \epsilon \mid \mathcal{F}_{t-t_0}\right) &\le \epsilon + \mathbb{P}_{u,v}\left(|M_2(t)| > \epsilon, \langle M_2\rangle_t \le \epsilon^3 \mid \mathcal{F}_{t-t_0}\right) \\
&\le 2\epsilon
\end{aligned} \tag{4.9}
$$

by a standard martingale argument (note that working with $M_2(t)$ conditionally on $\mathcal{F}_{t-t_0}$ poses no difficulties). This together with (4.4) and (4.5) shows that for $t > t_0(\epsilon)$, $0 < \delta < \delta_0(\epsilon)$ and $\epsilon < 1/4$,

(4.10)

$$
\begin{aligned}
&\mathbb{P}_{u,v}\left(|u_t(0) - u| \le 2\epsilon, v_t(0) \le \epsilon\right) \\
&\quad \ge \mathbb{P}_{u,v}\left(|M_2(t)| \le \epsilon, v_t(0) \le \epsilon\right) - \mathbb{P}_{u,v}\left(|M_1(t-t_0)| > \epsilon\right) \\
&\quad \ge (1-2\epsilon)\mathbb{P}_{u,v}\big(A_\delta(t)\big) - \mathbb{P}_{u,v}\left(|M_1(t-t_0)| > \epsilon, \langle M_1\rangle(t-t_0) \le \epsilon^3\right) - \mathbb{P}_{u,v}\left(\langle M_1\rangle_{t-t_0} > \epsilon^3\right) \\
&\quad \ge \frac{1}{2}\mathbb{P}_{u,v}\big(A_\delta(t)\big) - 2\epsilon,
\end{aligned}
$$

by a standard martingale inequality and (4.5). If $M(u) = \sum_k \int_0^u p_{t-s}(k,0)\big(\gamma u_s(k)v_s(k)\big)^{1/2} dB_s^k$ and $N(u) = \sum_k \int_0^u p_{t-s}(k,0)\big(\gamma u_s(k)v_s(k)\big)^{1/2} dW_s^k$ (L^2 convergent by Theorem 2.2(b)) ($u \le t$), then $\big(M(u), N(u)\big)$ may be written as $B(\tau_u)$ where B is a planar Brownian motion starting at the origin,

$$
\tau_u = \sum_k \int_0^u p_{t-s}(k,0)^2 \gamma u_s(k) v_s(k)\, ds, u \le t,
$$

and we may have enlarged the probability space to fill out the Brownian path B. Inverting (4.10) and using Theorem 2.2(b)(ii) with $\phi = \psi = 1_{\{0\}}$ we have for $t > t_0(\epsilon)$ and $0 < \delta < \delta_0(\epsilon)$,

$$
\begin{aligned}
\mathbb{P}_{u,v}\big(A_\delta(t)\big) &\le 2\mathbb{P}_{u,v}\left(|u_t(0) - u| \le 2\epsilon, v_t(0) \le \epsilon\right) + 4\epsilon \\
&\le 2\mathbb{P}_{u,v}\left(|B(\tau_t) + (0,v)| \le 3\epsilon\right) + 4\epsilon \\
&\le 2\mathbb{P}_{u,v}\left(|B(s) + (0,v)| \le 3\epsilon \text{ for some } s \le T\right) + 2\mathbb{P}_{u,v}(\tau_t > T) + 4\epsilon \\
&\le 2\mathbb{P}_{u,v}\left(|B(s) + (0,v)| \le 3\epsilon \text{ for some } s \le T\right) + (1/T)\gamma uv \int_0^\infty p_r(0,0)\, dr + 4\epsilon.
\end{aligned}
$$

If $\eta>0$ is fixed we may first fix T large enough and then ϵ sufficiently small so that for $\delta<\delta_0(\epsilon)$ and $t>t_0(\epsilon)$, $\mathbb{P}_{u,v}\big(A_\delta(t)\big)<\eta$. For $\delta>0$ fixed as above, since

$$(\langle u_{t-t_0},\tilde\phi_{-1}\rangle,\langle v_{t-t_0},\tilde\phi_{-1}\rangle)\xrightarrow{w}(\langle u_\infty,\tilde\phi_{-1}\rangle,\langle v_\infty,\tilde\phi_{-1}\rangle)\quad\text{as}\quad t\to\infty$$

by Theorem 1.4, we have

$$\mathbb{P}_{u,v}(\langle v_\infty,\tilde\phi_{-1}\rangle=0)\le\liminf_{t\to\infty}\mathbb{P}_{u,v}\big(A_\delta(t)\big)\le\eta.$$

This gives (4.3) and the proof of (a) is complete.
(b) If $\delta_k(j)=1(j=k)$, Corollaries 2.9 and 2.6 show that for $\theta>0$ and $k\in\mathbb{Z}^d$,

$$\begin{aligned}\mathbb{P}_{u,v}\left(\exp\left(-\theta\big(u_\infty(k)+v_\infty(k)\big)\right)\right)&=\mathbb{P}_{(\theta/2)\delta_k,(\theta/2)\delta_k}\left(\exp\{-(u+v)(\langle\tilde u_\infty,1\rangle+\langle\tilde v_\infty,1\rangle)\right.\\&\qquad\left.+i(u-v)(\langle\tilde u_\infty,1\rangle-\langle\tilde v_\infty,1\rangle)\}\right)\\&=\mathbb{P}_{\delta_k,\delta_k}\left(\exp\{-(u+v)(\theta/2)(\langle\tilde u_\infty,1\rangle+\langle\tilde v_\infty,1\rangle)\right.\\&\qquad\left.+i(u-v)(\theta/2)(\langle\tilde u_\infty,1\rangle-\langle\tilde v_\infty,1\rangle)\}\right).\end{aligned}$$

Differentiate both sides twice with respect to θ and let $\theta\downarrow0$ to see that

$$\begin{aligned}\mathbb{P}_{u,v}\left(\big(u_\infty(k)+v_\infty(k)\big)^2\right)&=\mathbb{P}_{\delta_k,\delta_k}\left(((u+v)^2/4)(\langle\tilde u_\infty,1\rangle+\langle\tilde v_\infty,1\rangle)^2-\big((u-v)^2/4\big)\right.\\&\quad\left.\times(\langle\tilde u_\infty,1\rangle-\langle\tilde v_\infty,1\rangle)^2\right)-i\big((u^2-v^2)/2\big)\mathbb{P}_{\delta_k,\delta_k}(\langle\tilde u_\infty,1\rangle^2-\langle\tilde v_\infty,1\rangle^2).\end{aligned}$$

This shows that $\mathbb{P}_{\delta_k,\delta_k}\big(\langle\tilde u_\infty,1\rangle^2-\langle\tilde v_\infty,1\rangle^2\big)=0$ and allows us to simplify the right side to give

$$\mathbb{P}_{u,v}\left(\big(u_\infty(k)+v_\infty(k)\big)^2\right)=(u^2+v^2)\mathbb{P}_{\delta_k,\delta_k}\big(\langle\tilde u_\infty,1\rangle\langle\tilde v_\infty,1\rangle\big)+uv\mathbb{P}_{\delta_k,\delta_k}\left(\langle\tilde u_\infty,1\rangle^2+\langle\tilde v_\infty,1\rangle^2\right).\tag{4.11}$$

Recall from Theorem 2.2(d) that under $\mathbb{P}_{\delta_k,\delta_k}$, $\langle\tilde u_t,1\rangle$ is a continuous martingale such that

$$\mathbb{P}_{\delta_k,\delta_k}\left(\langle\tilde u_t,1\rangle^2\right)=1+\gamma\int_0^t\sum_j p_s(k,j)^2\,ds=1+\gamma\int_0^t p_{2s}(k,k)\,ds\uparrow1+\frac{\gamma}{2}g_\infty(k,k)\quad\text{as}\quad t\to\infty.$$

Hence it is also L^2-bounded and as $\sup_t\langle\tilde u_t,1\rangle^2+\langle\tilde v_t,1\rangle^2$ is integrable by Doob's maximal inequality, the obvious uniform integrability allows us to deduce from (4.11) that

$$\begin{aligned}\mathbb{P}_{u,v}\left(\big(u_\infty(k)+v_\infty(k)\big)^2\right)&=\lim_{t\to\infty}(u^2+v^2)\mathbb{P}_{\delta_k,\delta_k}(\langle\tilde u_t,1\rangle\langle\tilde v_t,1\rangle)+uv\mathbb{P}_{\delta_k,\delta_k}(\langle\tilde u_t,1\rangle^2+\langle\tilde v_t,1\rangle^2)\\&=\lim_{t\to\infty}(u^2+v^2)+2uv\left(1+\gamma\int_0^t p_{2s}(k,k)\,ds\right)\quad\text{(by the above}\\&\qquad\qquad\text{and Theorem 2.2(b)(iii))}\\&=(u+v)^2+\gamma uvg_\infty(k,k).\end{aligned}\tag{4.12}$$

Theorem 2.2(b)(ii) and (iii) imply

$$\lim_{t\to\infty}\mathbb{P}_{u,v}\left(\left(u_t(k)+v_t(k)\right)^2\right)=\lim_{t\to\infty}u^2+v^2+2\gamma uv\int_0^t\sum_j p_{t-r}(j,k)^2\,dr+2uv$$
$$=(u+v)^2+\gamma uv g_\infty(k,k)$$
$$=\mathbb{P}_{u,v}\left(\left(u_\infty(k)+v_\infty(k)\right)^2\right)\quad\text{(by (4.12))}.$$

This together with the weak convergence of $u_t(k)+v_t(k)$ to $u_\infty(k)+v_\infty(k)$ shows that $\{\left(u_n(k)+v_n(k)\right)^2 : n\in\mathbb{N}\}$ is uniformly integrable for each k. This implies that

$$\mathbb{P}_{u,v}\big(u_\infty(j)u_\infty(k)\big)=\lim_{n\to\infty}\mathbb{P}_{u,v}\big(u_n(j)u_n(k)\big)$$
$$=u^2+\lim_{n\to\infty}\int_0^n\gamma uv\sum_\ell p_{n-r}(\ell,j)p_{n-r}(\ell,k)\,dr\quad\text{(Theorem 2.2.(b)(ii))}$$
$$=u^2+\frac{\gamma uv}{2}g_\infty(j,k).$$

Similarly we get $\mathbb{P}_{u,v}\big(u_\infty(j)v_\infty(k)\big)=uv$ and $\mathbb{P}_{u,v}\big(v_\infty(j)v_\infty(k)\big)=v^2+(\gamma uv/2)g_\infty(j,k)$. ∎

5. Some Ergodic Theorems and Open Problems.

If ξ is transient (and so (u_t,v_t) exhibits co-existence of types in equilibrium), it is easy to establish ergodicity under $\mathbb{P}_{u,v}$ by a second moment calculation as we now show. Let $I_T(K)=T^{-1}\int_0^T u_s(k)\,ds$, and $J_T(k)=T^{-1}\int_0^T v_s(k)\,ds$ for $k\in\mathbb{Z}^d$, $T>0$. A simple application of the Markov property and Theorem 2.2(b)(ii),(iii) shows that

$$\mathbb{P}_{u,v}\big(u_s(k)u_t(k)\big)=u^2+(\gamma uv/2)\int_{|t-s|}^{t+s}p_r(k,k)\,dr\quad k\in\mathbb{Z}^d,s,t\geq 0. \tag{5.1}$$

We work with respect to $\mathbb{P}_{u,v}$ throughout this section.

Proposition 5.1. Assume $g_\infty(k,k)<\infty$ for all $k\in\mathbb{Z}^d$.

(a) As $T\to\infty$, $I_T(k)\xrightarrow{L^2}u$ and $J_T(k)\xrightarrow{L^2}v$ for all $k\in\mathbb{Z}^d$.

(b) If for some $\eta>0$, $h(k,T)=\int_T^\infty p_r(k,k)\,dr\leq cT^{-\eta}$, then

$$\lim_{T\to\infty}I_T(k)=u\quad\text{and}\quad\lim_{T\to\infty}J_T(k)=v\quad\mathbb{P}_{u,v}\text{-a.s.}$$

Proof. From (5.1) we obtain

$$\mathbb{P}_{u,v}\left(I_T(k)^2\right)=T^{-2}\int_0^T\int_0^T\mathbb{P}_{u,v}\big(u_s(k)u_t(k)\big)\,ds\,dt=u^2+\gamma uv(2T^2)^{-1}\int_0^T\int_0^T\int_{|t-s|}^{t+s}p_r(k,k)\,dr\,ds\,dt. \tag{5.2}$$

Therefore, if $h(k,T)$ is as in (b),

$$\begin{aligned}
\mathbb{P}_{u,v}\left((I_T(k)-u)^2\right) &\le \gamma uv T^{-2}\int_0^T\int_0^t h(k,|t-s|)\,ds\,dt \\
&\le \gamma uv T^{-2}\int_0^T\int_{(t-\sqrt{T})^+}^t g_\infty(k,k)\,ds\,dt + \gamma uv T^{-2}\int_0^T\int_0^{(t-\sqrt{T})^+} h(k,\sqrt{T})\,ds\,dt \\
&\le \gamma uv g_\infty(k,k) T^{-1/2} + \gamma uv h(k,T^{1/2}).
\end{aligned} \tag{5.3}$$

Let $T\to\infty$ in the above to prove (a).

For (b) note that the hypothesis implies the right side of (5.3) is summable in n if we set $T=n^\rho$ and ρ is sufficiently large. The Borel-Cantelli Lemma shows that $\lim_{n\to\infty} I_{n^\rho}(k)=u$ a.s., and an easy interpolation argument completes the proof for I_T. The same derivation is valid for J_T. ∎

Remark 5.2. The hypotheses of both (a) and (b) are satisfied when ξ is a simple symmetric random walk on $\mathbb{Z}^d$ and $d\ge 3$. In this case the local central limit for the discrete time random walk (e.g. Lawler (1991), p.14) gives

$$\lim_{t\to\infty} t^{d/2}p_t(k,k) = (d/2\pi)^{d/2} \quad \forall k\in\mathbb{Z}^d \quad \forall d\in\mathbb{N}, \tag{5.4}$$

and so the hypothesis of (b) holds for $d\ge 3$ with $\eta=(d/2)-1$.

The long term dynamics of (u_t,v_t) in the recurrent case appear to be quite interesting but we have more questions than answers. To illustrate the problems, we focus on the case when ξ is simple symmetric random walk on $\mathbb{Z}^d$ and $d\le 2$.

Proposition 5.3. Let ξ be simple symmetric random walk on $\mathbb{Z}^d$ and $u,v>0$.

(a) If $d=1$ $\lim_{T\to\infty}\mathbb{P}_{u,v}\left(I_T(k)^2\right)=\lim_{T\to\infty}\mathbb{P}_{u,v}\left(J_T(k)^2\right)=\infty \quad \forall k\in\mathbb{Z}$.

(b) If $d=2$ $\lim_{T\to\infty}\mathbb{P}_{u,v}\left(I_T(k)^2\right)=u^2+\gamma uv(\ln 2)\pi^{-1}$ and $\lim_{T\to\infty}\mathbb{P}_{u,v}\left(J_T(k)^2\right)=v^2+\gamma uv(\ln 2)\pi^{-1}$ $\forall k\in\mathbb{Z}^2$.

Proof. (b) (5.4) and an easy estimate (using the calculation below) allows us to use (5.2) to conclude

$$\begin{aligned}
\lim_{T\to\infty} \mathbb{P}_{u,v}\left(I_T(k)^2\right) &= u^2 + \lim_{T\to\infty} \gamma uv T^{-2} \int_0^T \int_0^t \left(\int_{t-s}^{t+s} \left(\pi(r+1)\right)^{-1} dr \right) ds\, dt \\
&= u^2 + \lim_{T\to\infty} \gamma uv (\pi T^2)^{-1} \int_0^T \int_0^t \ln(1+t+s) - \ln(1+t-s)\, ds\, dt \\
&= u^2 + \lim_{T\to\infty} \gamma uv (\pi T^2)^{-1} \int_0^T 2t \ln\left(\frac{1+2t}{1+t}\right) + \ln(1+2t) - 2\ln(1+t)\, dt \\
&= u^2 + \lim_{T\to\infty} \gamma uv (\pi T^2)^{-1} \int_0^T 2t \ln\left(\frac{1+2t}{1+t}\right) dt \\
&= u^2 + \gamma uv (\ln 2)\pi^{-1}.
\end{aligned}$$

(a) Use the bound (from (5.4)) $p_r(k,k) \geq c(r+1)^{-1/2}$ and argue as above. ∎

Remark 5.4. Consider the $d = 2$ case. In spite of (b) above, $I_T(k)$ does not converge in L^2 as $T \to \infty$. To see this we can use (5.1) and (5.4) to see that if $T_1 = T_1(T_2)$ increases sufficiently quickly with T_2 (we certainly need $T_1 \gg T_2$), then

$$\lim_{T_2\to\infty} \mathbb{P}_{u,v}\left(I_{T_1}(k) I_{T_2}(k)\right) = u^2,$$

that is I_{T_1} and I_{T_2} are asymptotically uncorrelated. We omit this calculation. It then follows from Proposition 5.3(b) that

$$\begin{aligned}
\lim_{T_2\to\infty} \mathbb{P}_{u,v}\left(\left(I_{T_1(T_2)}(k) - I_{T_2}(k)\right)^2\right) &= \lim_{T_2\to\infty} \mathbb{P}_{u,v}\left(I_{T_1(T_2)}(k)^2 + I_{T_2}(k)^2\right) - 2\mathbb{P}_{u,v}\left(I_{T_1(T_2)}(k) I_{T_2}(k)\right) \\
&= 2\gamma uv(\ln 2)\pi^{-1} > 0,
\end{aligned}$$

and so $\{I_T(k)\}$ is not Cauchy in L^2 as $T \to \infty$.

Recall (Theorem 1.5) that under $\mathbb{P}_{u,v}$, $\left(u_t(k), v_t(k)\right) \xrightarrow{w} (B_T^1, B_T^2)$ as $t \to \infty$, where $T = \inf\{t : B_t^1 B_t^2 = 0\}$ and (B^1, B^2) is a planar Brownian motion starting at (u,v). We suspect that $\frac{1}{T}\int_0^T u_s(k)\, ds$ fails to converge because long stretches of time where k is in a large block dominated by the "u population" are followed by even longer stretches of time where k is in a larger block dominated by the v population (and the $u_t(k)$ values are negligible), etc. This picture of "alternating types" at a fixed site is consistent with the following simple result.

Proposition 5.5. Assume ξ is simple symmetric random walk on $\mathbb{Z}^2$ and $u, v > 0$. Under $\mathbb{P}_{u,v}$, $u_t(k)$ and $v_t(k)$ do not converge in probability as $t \to \infty$ $\forall k \in \mathbb{Z}^d$.

Proof. We may set $k = 0$. The fact that a.s. convergence fails is a simple consequence of Fatou's lemma, Proposition 5.3(b) and the fact that

$$\mathbb{P}_{u,v}\left(u_\infty(0)^2\right) = P_{u,v}^0\left((B_T^1)^2\right) = \mathbb{P}_{u,v}(T) = \infty. \tag{5.5}$$

For convergence in probability, the argument is only slightly more involved.

Suppose that $u_t(0)$ converges in probability to $u_\infty(0)$ under $\mathbb{P}_{u,v}$. Recall from Theorem 1.4 that

$$\mathbb{P}_{u,v}\big(u_\infty(0)\big) = u. \tag{5.6}$$

We claim there is a sequence $t_k \to \infty$ such that

$$\varliminf_{k\to\infty} I_{t_k}(0) \ge u_\infty(0) \quad \text{a.s.} \tag{5.7}$$

To see this, first note that for each M in $\mathbb{N}$, $|u_\infty(0) - u_t(0)|1(u_t(0) \le M) \le u_\infty(0) + M$ and so by dominated convergence (use (5.6)) we have $\lim_{t\to\infty} \mathbb{P}_{u,v}\big(|u_\infty(0) - u_t(0)|1(u_t(0) \le M)\big) = 0$ and therefore

$$\lim_{T\to\infty} \mathbb{P}_{u,v}\left(T^{-1}\int_0^T |u_\infty(0) - u_t(0)|1(u_t(0) \le M)\,dt\right) = 0 \quad \text{for each} \quad M \in \mathbb{N}.$$

As we have

$$\lim_{T\to\infty} \mathbb{P}_{u,v}\left(T^{-1}\int_0^T 1(|u_\infty(0) - u_t(0)| > 1)\,dt\right) = 0,$$

if $M_k \uparrow \infty (M_k \in \mathbb{N})$, we may choose $t_k \uparrow \infty$ such that for a.a. ω,

$$\lim_{k\to\infty} t_k^{-1}\int_0^{t_k} 1(|u_\infty(0) - u_t(0)| > 1) + |u_\infty(0) - u_t(0)|1(u_t(0) \le M_k)\,dt = 0. \tag{5.8}$$

Fix ω so that (5.8) is valid. Then

$$\begin{aligned}
\varliminf_{k\to\infty} I_{t_k}(0) &\ge \varliminf_{k\to\infty} t_k^{-1}\int_0^{t_k} u_t(0)1(u_t(0) \le M_k)\,dt \\
&= \varliminf_{k\to\infty} t_k^{-1}\int_0^{t_k} (u_t(0) - u_\infty(0))1(u_t(0) \le M_k)\,dt + u_\infty(0)t_k^{-1}\int_0^{t_k} 1(u_t(0) \le M_k)\,dt \\
&= u_\infty(0) - u_\infty(0)\varlimsup_{k\to\infty} t_k^{-1}\int_0^{t_k} 1(u_t(0) > M_k)\,dt \quad \text{(by (5.8))} \\
&\ge u_\infty(0) - u_\infty(0)\varlimsup_{k\to\infty}\left(1(u_\infty(0) > M_k - 1) + t_k^{-1}\int_0^{t_k} 1(|u_\infty(0) - u_t(0)| > 1)\,dt\right) \\
&= u_\infty(0) \quad \text{(by (5.8) again).}
\end{aligned}$$

This proves (5.7). Fatou's lemma now shows that

$$\varliminf_{k\to\infty} \mathbb{P}_{u,v}\left(I_{t_k}(0)^2\right) \geq \mathbb{P}_{u,v}\left(u_\infty(0)^2\right) = \infty \quad \text{by (5.5).}$$

This contradicts Proposition 5.3(b) and so convergence in probability must fail. ∎

We conjecture that in the setting of Proposition 5.5

$$\{u_t(0) : t \geq T\} = \{v_t(0) : t \geq T\} = (0,\infty) \quad \forall T > 0 \quad \text{a.s.}$$

The description of the large time dynamics when $d = 1$ remains completely unresolved.

6. A Stochastic Partial Differential Equation.

We now study solutions (u, v) of $(SPDE)_{u_0,v_0}$. Here are some additional state spaces for the solutions, which augment the spaces introduced in Section 1. Recall that $|f|_\lambda = \sup_x |f(x)e^{\lambda|x|}|$.

$$C_{rap} = \{f \in C(\mathbb{R}) : |f|_\lambda < \infty \quad \forall \lambda > 0\}, \quad \text{topologized by the metric}$$
$$d_{rap}(f,g) = \sum_{n=1}^{\infty} \big(|f-g|_{\mu_n} \wedge 1\big)2^{-n} \quad \text{where} \quad \mu_n \uparrow \infty.$$

Ω_{rap} is the space of $(C^+_{rap})^2$-valued paths with the compact-open topology.

$\{P_t : t \geq 0\}$ denotes the Brownian semigroup on the bounded measurable functions on $\mathbb{R}$ and $p_t(x)$ is the Brownian transition density.

$$C_{int} = \{f \in C(\mathbb{R}) : \langle |f|, 1\rangle < \infty\}, C_b = \{f \in C(\mathbb{R}) : \sup_x |f(x)| < \infty\},$$
$$C_p = \{f \in C(\mathbb{R}) : |f|_p < \infty\}, p \in \mathbb{R}.$$

Theorem 6.1. (a) If $u_0, v_0 \in C^+_{tem}$, there is a solution to $(SPDE)_{u_0,v_0}$, defined on some $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$.

(b) Let (u, v, W_1, W_2) be any solution to $(SPDE)_{u_0,v_0}$ on some $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ for a given $(u_0, v_0) \in (C^+_{tem})^2$.

(i) If $\phi \in C_{rap}$, then $\forall t > 0$

$$\langle u_t, \phi\rangle = \langle u_0, P_t\phi\rangle + \int_0^t \int P_{t-s}\phi(x)\big(\gamma u(s,x)v(s,x)\big)^{1/2} dW_1(s,x)$$

and

$$\langle v_t, \phi\rangle = \langle v_0, P_t\phi\rangle + \int_0^t \int P_{t-s}\phi(x)\big(\gamma u(s,x)v(s,x)\big)^{1/2} dW_2(s,x) \quad \text{a.s.,}$$

where both the stochastic integrals are square integrable.

(ii) $u(t,x) = P_t u_0(x) + \int_0^t \int p_{t-s}(y-x)\big(\gamma u(s,y)v(s,y)\big)^{1/2} dW_1(s,y)$

$$v(t,x) = P_t v_0(x) + \int_0^t \int p_{t-s}(y-x)\big(\gamma u(s,y)v(s,y)\big)^{1/2} dW_2(s,y) \quad \text{a.s. for each} \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R},$$

where both stochastic integrals are square integrable.

(iii) For all $s,t \in \mathbb{R}_+$, $x,y \in \mathbb{R}$, λ, $q > 0$,

$$\mathbb{P}\big(u(t,x)\big) = P_t u_0(x), \quad \mathbb{P}\big(v(t,x)\big) = P_t v_0(x), \quad \mathbb{P}\big(u(s,x)v(t,y)\big) = P_s u_0(x) P_t v_0(y), \quad \text{and}$$

$$\sup_{r \le t} \mathbb{P}\left(\int \big(u_r(x)^q + v_r(x)^q\big) e^{-\lambda|x|}\, dx\right) < \infty. \tag{6.1}$$

If $\phi, \psi : \mathbb{R} \to \mathbb{R}_+$ are measurable, then for all $s,t \in \mathbb{R}_+$,

$$\begin{aligned}&\mathbb{P}\big(\langle u_t, \phi\rangle\big) = \langle u_0, P_t\phi\rangle, \quad \mathbb{P}\big(\langle v_t, \psi\rangle\big) = \langle v_0, P_t\psi\rangle, \quad \text{and}\\ &\mathbb{P}\big(\langle u_s, \phi\rangle\langle v_t, \psi\rangle\big) = \langle u_0, P_s\phi\rangle\langle v_0, P_t\psi\rangle.\end{aligned}$$

(iv) If $\phi, \psi \in C^2_{rap}$, then

$$\langle u_t, \phi\rangle = \langle u_0, \phi\rangle + \int_0^t \langle u_s, \phi''/2\rangle\, ds + \int_0^t \int \phi(x)\big(\gamma u(s,x)v(s,x)\big)^{1/2}\, dW_1(s,x)$$

and

$$\langle v_t, \psi\rangle = \langle v_0, \psi\rangle + \int_0^t \langle v_s, \psi''/2\rangle\, ds + \int_0^t \int \psi(x)\big(\gamma u(s,x)v(s,x)\big)^{1/2}\, dW_2(s,x),$$

where the stochastic integrals are orthogonal square integrable continuous $(\mathcal{F}_t)$-martingales and the Lebesgue integrals are of integrable total variation.

(c) Let (u,v,W_1,W_2) be as in (b) but with $(u_0,v_0) \in (C^+_{rap})^2$. Then $(u_\cdot, v_\cdot) \in \Omega_{rap}$ a.s. and

$$\sup_{r \le T} \mathbb{P}\left(\int \big(u(r,x)^q + v(r,x)^q\big) e^{\lambda|x|}\, dx\right) < \infty \quad \forall q, T, \lambda > 0. \tag{6.2}$$

(b)(i) holds for $\phi \in C_{tem}$ and (b)(iv) is valid for $\phi, \psi \in C^2_{tem}$.

(d) Let (u,v,W_1,W_2) be as in (b) but with $(u_0,v_0) \in (C^+_{int})^2$. Then $(u_t,v_t) \in (C^+_{int})^2$ $\forall t \ge 0$ a.s., (b)(i) holds for $\phi \in C_b$ and (b)(iv) holds for $\phi, \psi \in C^2_b$. In particular $\langle u_t, 1\rangle$ and $\langle v_t, 1\rangle$ are orthogonal square integrable continuous $(\mathcal{F}_t)$-martingales.

The proof is presented in the Appendix. Many of the ideas in Shiga (1994) are easily modified to the present context.

As for the discrete setting in Section 2, we introduce state spaces for our anticipated self-dual processes.

Notation. $F = \{(X,Y) \in C^+_{tem} \times C_{tem} : |Y| \le X \quad \text{on} \quad \mathbb{R}\}$.

$\tilde{F} = \{(X,Y) \in F : X \in C^+_{rap}\} \supset F_c = \{(X,Y) \in F : X \quad \text{has compact support}\}$.

Metrize F by $d_F\big((X,Y),(X',Y')\big) = d_{tem}(X,X') + d_{tem}(Y,Y')$ and similarly metrize $\tilde{F}$ using d_{rap}.

If $(\phi,\psi) \in \tilde{F}$, define $G_{\phi,\psi} : F \to \mathbb{C}$ by $G_{\phi,\psi}(X,Y) = \exp\{-\langle X,\phi\rangle + i\langle Y,\psi\rangle\}$.

It is easy to check that (F, d_F) and $(\tilde{F}, d_{\tilde{F}})$ are Polish spaces.

Lemma 6.2. If P and Q are probabilities on F such that $\int G_{\phi,\psi} dP = \int G_{\phi,\psi} dQ \quad \forall(\phi,\psi) \in F_c$, then $P = Q$.

Proof. If $\mathcal{A} = \{G_{\phi,\psi} : (\phi,\psi) \in F_c\}$ and $\mathcal{C}$ is the complex linear span of $\mathcal{A}$, then $\mathcal{C}$ is a complex algebra containing the constants and closed under complex conjugation. A monotone class theorem (e.g. see I.21, I.22 of Dellacherie and Meyer (1978)) reduces the problem to showing that $\sigma(\mathcal{A})$ (the minimal σ-field making functions in $\mathcal{A}$ measurable) is the Borel σ-field of F. For this it suffices to fix rational numbers s and t, and show that $(X,Y) \to \big(X(s),Y(t)\big)$ is $\sigma(\mathcal{A})$-measurable. Let $\{K_\epsilon(\cdot) : \epsilon \in (0,1]\}$ be a continuous approximate identity (as $\epsilon \downarrow 0$) with compact support. Then

$$\big(X(s),Y(t)\big) = \lim_{\epsilon\downarrow 0} \left(\langle X, K_\epsilon(\cdot - s) + K_\epsilon(\cdot - t)\rangle, \langle Y, K_\epsilon(\cdot - t)\rangle\right) - \left(\langle X, K_\epsilon(\cdot - t)\rangle, \langle Y, 0\rangle\right)$$

and $\big(K_\epsilon(\cdot - s) + K_\epsilon(\cdot - t), K_\epsilon(\cdot - t)\big)$ and $\big(K_\epsilon(\cdot - t), 0\big)$ are in F_c. The result follows easily. ∎

Just as in the discrete setting we let (u_t,v_t) and $(\tilde{u}_t,\tilde{v}_t)$ denote the coordinate variables on Ω_{tem} and Ω_{rap}, respectively (as well as arbitrary solutions of $SPDE$), and define

$$(X_t,Y_t) = (u_t + v_t, u_t - v_t) \in F \quad \text{and} \quad (\tilde{X}_t,\tilde{Y}_t) = (\tilde{u}_t + \tilde{v}_t, \tilde{u}_t - \tilde{v}_t) \in \tilde{F}.$$

If $(u_0,v_0) \in \big(C^+_{rap}\big)^2$, Theorem 6.1 allows us to view the law of a solution to $(SPDE)_{u_0,v_0}$ as a probability on Ω_{tem} or Ω_{rap}.

Theorem 6.3 (Mytnik (1996)). (a) If $(u_0,v_0) \in (C^+_{tem})^2$, there is a unique probability $\mathbb{P}_{u_0,v_0}$ on Ω_{tem} so that (u,v) has law $\mathbb{P}_{u_0,v_0}$ whenever (u,v,W_1,W_2) is a solution to $(SPDE)_{u_0,v_0}$ on some $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$.

(b) If $(u_0,v_0) \in \big(C^+_{tem}\big)^2$ and $(\tilde{u}_0,\tilde{v}_0) \in \big(C^+_{rap}\big)^2$, then

$$\mathbb{P}_{u_0,v_0}\left(\exp\{-\langle X_t,\tilde{X}_0\rangle + i\langle Y_t,\tilde{Y}_0\rangle\}\right) = \mathbb{P}_{\tilde{u}_0,\tilde{v}_0}\left(\exp\{-\langle X_0,\tilde{X}_t\rangle + i\langle Y_0,\tilde{Y}_t\rangle\}\right).$$

Remark. Mytnik's proof was for $u_0,v_0 \in C^+_{int}$ but trivially adapts to the C^+_{tem}-valued setting. The argument is very close to the proof of Theorem 2.4. Clearly (b) implies the uniqueness in

(a) by Lemma 6.2. (b) is established by a duality argument. The latter differs slightly from that given in Theorem 2.4 because test functions for X_t (or $\tilde{X}_s$) should be smooth and $\tilde{X}_s$ (or X_t) is not. One may use the heat semigroup P_ϵ to smooth these functions and then let $\epsilon \downarrow 0$.

If $G : (C^+_{tem})^2 \to \mathbb{R}$ is bounded and measurable let $\overline{P}_t G(u_0, v_0) = \mathbb{P}_{u_0,v_0}\big(G(u_t, v_t)\big)$ denote the semigroup associated with solutions of $(SPDE)$. To prove the Feller property of $\{\overline{P}_t\}$ we will use the following slight modification of Lemma 6.3(ii) of Shiga (1994).

Lemma 6.4. Let $\{P_n : n \in \mathbb{N}\}$ be a sequence of probabilities on C_{tem}. Suppose $\forall\lambda > 0$ there are C, $p > 0$ and $\alpha > 1$ such that

$$\sup_n P_n\big(|u(x) - u(x')|^{2p}\big) \le Ce^{\lambda|x|}|x - x'|^\alpha \quad \text{for all} \quad |x - x'| \le 1, \quad \text{and}$$
$$\sup_n P_n\big(|u(0)|\big) \le C.$$

Then $\{P_n\}$ is tight.

Corollary 6.5. (a) $\overline{P}_t : C_b\big((C^+_{tem})^2\big) \to C_b\big((C^+_{tem})^2\big)$ for each $t \ge 0$.

(b) Let (u, v, W_1, W_2) solve $(SPDE)_{u_0,v_0}$ on some $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ and let T be an a.s. finite $(\mathcal{F}_t)$-stopping time. For any bounded measurable f on $(C^+_{tem})^2$

$$\mathbb{P}\left(f(u_{T+t}, v_{T+t}) \mid \mathcal{F}_T\right) = \overline{P}_t f(u_T, v_T) \quad \mathbb{P}\text{-a.s.} \quad \text{for all} \quad t \ge 0.$$

Proof. (a) Let $(u_0^n, v_0^n) \to (u_0, v_0)$ in $(C^+_{tem})^2$, $t > 0$ and set $P_n(A) = \mathbb{P}_{u_0^n,v_0^n}\big((u_t, v_t) \in A\big)$. We must show the weak convergence of P_n to $\mathbb{P}_{u_0,v_0}\big((u_t, v_t) \in A\big)$. Theorem 6.3(b) implies that for $(\phi, \psi) \in \tilde{F}$

$$\lim_{n\to\infty} \mathbb{P}_{u_0^n,v_0^n}\big(G_{\phi,\psi}(u_t + v_t, u_t - v_t)\big) = \mathbb{P}_{u_0,v_0}\big(G_{\phi,\psi}(u_t + v_t, u_t - v_t)\big)$$

and, as this class of functions is a determining class on $(C^+_{tem})^2$ by Lemma 6.2, it suffices to prove that $\{P_n\}$ is tight. For this it suffices to show $\{P_n(u \in \cdot)\}$ is tight on C^+_{tem} and here we use Lemma 6.4. The last condition of Lemma 6.4 holds by Theorem 6.1(b)(iii). The first condition is implicit in the derivation of Theorem 6.1(a) but this time we sketch the details. By Theorem 6.1(b)(ii), an easy estimate on $|P_t u_0(x) - P_t u_0(x')|$ and the Burkholder-Davis-Gundy inequality this reduces to showing that for $\lambda > 0$ there are $C, p \ge 1$ and $\alpha > 1$ so that

(6.3)

$$\sup_n \mathbb{P}_{u_0^n,v_0^n}\left(\left(\int_0^t\int \big(p_{t-s}(y - x) - p_{t-s}(y - x')\big)^2\gamma u(s, y)v(s, y)\, dy\, ds\right)^p\right) \le Ce^{\lambda|x|}|x - x'|^\alpha$$
$$\text{for} \quad |x - x'| \le 1.$$

By Jensen's inequality, the left side of (6.2) is bounded by (let $\lambda' = \frac{3}{4}\lambda$)
(6.4)

$$\gamma^p \sup_n \mathbb{P}_{u_0^n, v_0^n} \left(\int_0^t \int \big(p_{t-s}(y-x) - p_{t-s}(y-x')\big)^2 \big(u(s,y)^{2p} + v(s,y)^{2p}\big)\, dy\, ds \right)$$
$$\left(\int_0^t \int \big(p_{t-s}(y-x) - p_{t-s}(y-x')\big)^2\, dy\, ds \right)^{p-1}$$
$$\leq C \sup_n \left[\mathbb{P}_{u_0^n, v_0^n} \left(\int_0^t \int \big(u(s,y)^{8p} + v(s,y)^{8p}\big) e^{-4\lambda'|y|}\, dy\, ds \right) \right]^{1/4}$$
$$\left[\int_0^t \int e^{\frac{4}{3}\lambda|y'|} \big(p_{t-s}(y-x) - p_{t-s}(y-x')\big)^{8/3}\, dy\, ds \right]^{3/4} \times |x-x'|^{p-1}.$$

In the last line used Lemma 6.2 of Shiga (1994) to bound the last term in (6.4) by $C|x-x'|^{p-1}$ and used Hölder's inequality on the first term. Using Lemma 6.2 of Shiga (1994) again, for $|x-x'| \leq 1$ we get

$$\int_0^t \int e^{\frac{4}{3}\lambda'|y|} |p_{t-s}(y-x) + p_{t-s}(y-x')|^{8/3}\, dy\, ds$$
$$\leq \int_0^t (t-s)^{-5/6} \int e^{\lambda|y|} \big(p_{t-s}(y-x) + p_{t-s}(y-x')\big)\, dy\, ds \leq C e^{\lambda|x|}.$$

A Gronwall argument (see (6.16) of Shiga (1994)), as in the derivation of (6.1), shows that the first factor on the right side of (6.4) is finite. Using these estimates in (6.4), we see that (6.3) holds with $\alpha = p-1$ and taking $p > 2$, we are done.

(b) This is a consequence of Theorem 6.3, whose proof actually shows the uniqueness of the analogous martingale problem (as for Theorem 2.4), and standard arguments (see Theorems 4.4.2 and 4.4.6 of Ethier and Kurtz (1986)). ∎

As in the lattice case, the duality relation in Theorem 6.3(b) reduces the longtime behaviour of (u_t, v_t) with constant initial conditions to the longtime behaviour of $\big(\langle u_t, 1\rangle, \langle v_t, 1\rangle\big)$ with rapidly decreasing initial conditions.

Theorem 6.6. Let $u_0, v_0 \in C^+_{rap}$. Then $\lim_{t\to\infty} \langle u_t, 1\rangle \equiv \langle u_\infty, 1\rangle$ and $\lim_{t\to\infty} \langle v_t, 1\rangle = \langle v_\infty, 1\rangle$ exist $\mathbb{P}_{u_0,v_0}$-a.s. and $\langle u_\infty, 1\rangle\langle v_\infty, 1\rangle = 0$ $\mathbb{P}_{u_0,v_0}$-a.s.

Proof. The existence of the a.s. limits follows from Theorem 6.1(d) and the martingale convergence theorem. The proof of the last assertion (no co-existence) proceeds as for the recurrent lattice case and so we only sketch the parts which are different from the proof of Theorem 1.2(b). Set $\gamma = 1$ to simplify the notation. Let $A_t = \int_0^t \langle u_s, v_s\rangle\, ds$. As in the proof of Theorem 1.2(b) we use Theorem 6.1(b)(ii) to write

$$A_T = A_T^{(1)} + N_T \tag{6.5}$$

where $A_T^{(1)} = \int_0^T \langle P_t u_0, P_t v_0\rangle\, dt$ and $(N_t, t \le T)$ is a continuous martingale satisfying

$$\langle N\rangle_T \le c\big(\langle u_T^*, 1\rangle^2 + \langle v_T^*, 1\rangle^2\big) g_{2T}(0)^2 A_T. \tag{6.6}$$

Here $\langle u_T^*, 1\rangle = \sup_{t\le T}\langle u_t, 1\rangle$, similarly for $\langle v_T^*, 1\rangle$, and $g_{2T}(x) = \int_0^{2T} p_s(x)\, ds$. As in the discrete case, some care is needed in the derivation of these formulae. A two-parameter stochastic Fubini theorem is required (Theorem 2.6 of Walsh (1986) may be used). To verify the integrability hypothesis of this Fubini theorem one needs to use (6.2) for appropriate q and $\lambda = 0$.

If $M = M(u_0, v_0)$ is chosen sufficiently large so that $\langle u_0, 1_{[-M/2,M/2]}\rangle \ge \frac{1}{2}\langle u_0, 1\rangle$, and similarly for v_0, then an application of Chapman-Kolmogorov leads to

$$A_T^{(1)} = \int\int g_{2T}(y-w)u_0(y)v_0(w)\, dy\, dw \ge g_{2T}(M)\frac{1}{4}\langle u_0, 1\rangle\langle v_0, 1\rangle.$$

If $T > M^2$,

$$g_{2T}(M) \ge e^{-1/2}\int_T^{2T}(2\pi t)^{-1/2}\, dt \ge e^{-1/2}\left[g_{2T}(0) - \frac{1}{\sqrt{2}}g_{2T}(0)\right] = c_0 g_{2T}(0),$$

and so for a universal constant $c_1 > 0$, we have

$$A_T^{(1)} \ge c_1 g_{2T}(0)\langle u_0, 1\rangle\langle v_0, 1\rangle \quad \text{for} \quad T > M(u_0, v_0)^2. \tag{6.7}$$

The proof now proceeds exactly as that of Theorem 1.2(b) using (6.6), (6.7) and the fact that $\lim_{t\to\infty} g_t(0) = \infty$, in place of (3.6), (3.9) and (3.8).

We recall the notation introduced prior to Theorem 1.8. If $(\phi, \psi) \in \tilde{F}$ define $H_{\phi,\psi} : (M_{tem})^2 \to \mathbb{C}$ by $H_{\phi,\psi}(\mu,\nu) = \exp\{-\langle \mu+\nu, \phi\rangle + i\langle \mu-\nu, \psi\rangle\}$. Recall also we may consider C_{tem}^+ as a subset of M_{tem} and so $H_{\phi,\psi}$ is defined on $(C_{tem}^+)^2$ by the above.

Lemma 6.7. Let $\{P_n\}$ be a sequence of probabilities on $(M_{tem})^2$. If

$$\sup_n \int \langle \mu+\nu, \phi_{-\lambda}\rangle dP_n(\mu,\nu) < \infty \quad \forall\lambda > 0 \tag{6.8}$$

and for each (ϕ,ψ) in F_c, $\lim_{n\to\infty} P_n(H_{\phi,\psi})$ exists and is finite, then $P_n \xrightarrow{w} P_\infty$ in $(M_{tem})^2$ and $\lim_{n\to\infty} P_n(H_{\phi,\psi}) = P_\infty(H_{\phi,\psi})\ \forall(\phi,\psi) \in F_c$.

Proof. It is straightforward to show that (6.8) implies $\{P_n : n \in \mathbb{N}\}$ is relatively compact in the weak topology. An argument smilar to, but simpler than, the proof of Lemma 6.1 shows that $\{H_{\phi,\psi} : (\phi,\psi) \in F_c\}$ is a determining class on $(M_{tem})^2$. The result follows. ∎

Proof of Theorem 1.8. We apply the previous lemma. Theorem 6.1(b)(iii) shows that if $\lambda > 0$, then

$$\sup_t \mathbb{P}_{u,v}\big(\langle u_t + v_t, \phi_{-\lambda}\rangle\big) = \sup_t (u+v) \int P_t\phi_{-\lambda}(x)\,dx = (u+v)\int \phi_{-\lambda}(x)\,dx < \infty.$$

Let $(\phi,\psi) \in F_c$. Apply Theorem 6.3(b) with $(\tilde{u}_0, \tilde{v}_0) = (\phi+\psi, \phi-\psi)/2$ to see that

$$\begin{aligned}\lim_{t\to\infty} \mathbb{P}_{u,v}\big(H_{\phi,\psi}(u_t,v_t)\big) &= \lim_{t\to\infty} \mathbb{P}_{\tilde{u}_0,\tilde{v}_0}\Big(\exp\big(-(u+v)\langle 1, \tilde{u}_t + \tilde{v}_t\rangle + i(u-v)\langle 1, \tilde{u}_t - \tilde{v}_t\rangle\big)\Big) \\ &= \mathbb{P}_{\tilde{u}_0,\tilde{v}_0}\Big(\exp\big(-(u+v)\langle 1, \tilde{u}_\infty + \tilde{v}_\infty\rangle + i(u-v)\langle 1, \tilde{u}_\infty - \tilde{v}_\infty\rangle\big)\Big).\end{aligned}$$

In the last line $\langle 1, \tilde{u}_\infty \pm \tilde{v}_\infty\rangle = \lim_{t\to\infty}\langle 1, \tilde{u}_t \pm \tilde{v}_t\rangle$ exist a.s. by the martingale convergence theorem (recall Theorem 6.1(d)). The previous lemma now establishes the weak convergence of $\mathbb{P}_{u,v}\big((u_t,v_t) \in \cdot\big)$ in $(M_{tem})^2$ as $t \to \infty$. The identification of the weak limit now proceeds just as in the proof of Theorem 1.5. Theorem 6.6 is used in place of Theorem 1.2(b). ∎

7. Appendix: Proofs of Theorems 2.2 and 6.1.

Proof of Theorem 2.2.(a). Skorokhod's Peano existence theorem shows that on some $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ there are independent $(\mathcal{F}_t)$-Brownian motions $\{B^k, W^k : k \in S_n\}$ and solutions $(u_t^n(k), v_t^n(k) : k \in \mathbb{Z}^d, t \geq 0)$ of

$$(LS)_n \qquad \begin{aligned} &u_t^n(k) = u_0(k),\ v_t^n(k) = v_0(k) \quad \text{for} \quad k \in S_n^c, t \geq 0 \\ &u_t^n(k) = u_0(k) + \int_0^t u_s^n Q(k)\,ds + \int_0^t \gamma |u_s^n(k) v_s^n(k)|^{1/2}\,dB_s^k,\ k \in S_n, t \geq 0 \\ &v_t^n(k) = v_0(k) + \int_0^t v_s^n Q(k)\,ds + \int_0^t \gamma |u_s^n(k) v_s^n(k)|^{1/2}\,dW_s^k,\ k \in S_n, t \geq 0.\end{aligned}$$

Note that by writing

$$u_s^n Q(k) = \sum_{j\in S_n} u_s^n(j) q_{jk} + \sum_{j \in S_n^c} u_0(j) q_{jk}$$

we see that $(LS)_n$ is a finite dimensional s.d.e. (H_1), (H_2) and the fact that $u_0, v_0 \in M_{tem}$ show the infinite series in the above is absolutely convergent.

We claim that $u_t^n(k) \geq 0$ and $v_t^n(k) \geq 0$ $\forall k \in \mathbb{Z}^d$, $\forall t \geq 0$ a.s. Note that the local time at 0 of $u_\cdot^n(k)$ is

$$L_t^0 = \lim_{\epsilon\downarrow 0} \int_0^t 1\big(0 < u_s^n(k) \leq \epsilon\big)\gamma |u_s^n(k)||v_s^n(k)|\,ds\epsilon^{-1} \leq \lim_{\epsilon\downarrow 0}\int_0^t 1\big(0 < u_s^n(k) \leq \epsilon\big)|v_s^n(k)|\,ds = 0.$$

Tanaka's formula implies that

$$\sum_{k\in S_n} u_t^n(k)^- = M_t + \int_0^t a(s)\,ds$$

where M is a continuous local martingale and

$$\begin{aligned} a(s) = &\sum_{k\in S_n} u_s^n(k)^- q_{kk} - \sum_{k\in S_n}\sum_{j\in S_n, j\neq k} 1\big(u_s^n(k)<0\big)u_s^n(j)q_{jk} \\ &\qquad - \sum_{k\in S_n} 1\big(u_s^n(k)<0\big)\Bigg(\sum_{j\in S_n^c} u_0(j)q_{jk}\Bigg) \\ \leq &\sum_{k\in S_n}\sum_{j\in S_n} u_s^n(j)^- q_{jk} \leq 0. \end{aligned}$$

The last inequality is clear because for each j in S_n, $\sum\limits_{k\in S_n} q_{jk} \leq \sum\limits_k q_{jk} = 0$. The previous inequality is seen by considering the cases $u_s^n(k) < 0$ and $u_s^n(k) \geq 0$ separately. This shows $\sum\limits_{k\in S_n} u_t^n(k)^-$ is a non-negative supermartingale starting at zero and hence is identically zero a.s. The same argument is valid for v_t^n. This proves the non-negativity claim.

Let $T_n^N = \inf\{t : \langle u_t^n + v_t^n, 1_{S_n}\rangle > N\}$. Clearly $T_n^N \uparrow \infty$ as $N \to \infty$ for each n. If $\phi_s(j) = p_{t-s}(j,k)$ for $s \leq t$ (t, k fixed) then $\phi_s \in M_{rap}$ (by (H_2)) and $\dot{\phi}_s(j) = -Q\phi_s(j)$ (by (H_0)) is bounded and continuous. Using $(LS)_n$ and Itô's Lemma we have for $s \leq t$

$$\begin{aligned} \langle u_s^n, \phi_s\rangle &= \langle u_0, 1_{S_n^c}\phi_s\rangle + \langle u_0, 1_{S_n}\phi_0\rangle + \int_0^s \langle u_r^n Q, 1_{S_n}\phi_r\rangle + \langle u_r^n, 1_{S_n}\dot{\phi}_r\rangle\,dr \\ &\qquad + \sum_{j\in S_n}\int_0^s \phi_r(j)\big(\gamma u_r^n(j)v_r^n(j)\big)^{1/2}\,dB_r^j \\ &= \langle u_0, 1_{S_n^c}(\phi_s - \phi_0)\rangle + \langle u_0, \phi_0\rangle + \int_0^s \langle u_r^n, Q1_{S_n}\phi_r - 1_{S_n}Q\phi_r\rangle\,dr + N_s^{n,u}(\phi), \end{aligned}$$

where $N_s^{n,u}(\phi)$ is defined to be the sum of the stochastic integrals in the previous line. Write

$$\langle u_r^n, Q1_{S_n}\phi_r - 1_{S_n}Q\phi_r\rangle = \beta^n(r) - \alpha^n(r)$$

where

$$\beta^n(r) = \sum_{j\notin S_n}\sum_{j'\in S_n} u_r^n(j)q_{jj'}\phi_r(j') = \sum_{j\notin S_n}\sum_{j'\in S_n} u_0(j)q_{jj'}\phi_r(j') \geq 0$$

and

$$\alpha^n(r) = \sum_{j\in S_n}\sum_{j'\notin S_n} u_r^n(j)q_{jj'}\phi_r(j') \geq 0.$$

If $t'_N = t \wedge T_n^N$, we therefore have

$$\langle u^n_{t'_N}, \phi_{t'_N}\rangle = \langle u_0, p_t(\cdot,k)\rangle + N^{n,u}_{t'_N}(\phi) + \langle u_0, 1_{S_n^c}(\phi_{t'_N} - \phi_0)\rangle + \int_0^{t'_N} \beta^n(r) - \alpha^n(r)\,dr. \tag{7.1}$$

By (H_1) and (H_2), for any $\lambda' > 0$ and $\overline{\lambda} > 0$ such that $\lambda'(2\overline{\lambda}) \le \lambda'$,

$$\begin{aligned} \sup_{s\le t}\langle u_0, 1_{S_n^c}\phi_s\rangle &\le c_{\overline{\lambda}} \sup_{s\le t} \sum_{j\in S_n^c} e^{\overline{\lambda}|j|} p_s(j,k) \\ &\le c_{\overline{\lambda}} e^{-\overline{\lambda}n} \sup_{s\le t} \sum_{j\in S_n^c} e^{2\overline{\lambda}|j|} p_s(j,k) \\ &\le c(t,\overline{\lambda}) e^{-\overline{\lambda}n} e^{\lambda'|k|}, \end{aligned} \tag{7.2}$$

for some $c(t,\overline{\lambda})$ which is increasing in t. We will implicitly assume all constants increase with t in this argument. Similarly for any $\lambda' > 0$ and $\overline{\lambda} > 0$ such that $\lambda'(\lambda'(2\overline{\lambda})) \le \lambda'$ we have

$$\begin{aligned} \sup_{r\le t}\beta^n(r) &\le c_{\overline{\lambda}} \sup_{s\le t} \sum_{j\notin S_n} \sum_{j'\in S_n} e^{\overline{\lambda}|j|} q_{jj'} p_s(j',k) \\ &\le e^{-\overline{\lambda}n} c_{\overline{\lambda}} \sup_{s\le t} \sum_{j\notin S_n} \sum_{j'\in S_n} e^{2\overline{\lambda}|j|} q_{jj'} p_s(j',k) \\ &\le c(t,\overline{\lambda},\lambda') e^{-\overline{\lambda}n} e^{\lambda'|k|} \quad \text{(by } (H_2) \text{ and } (H_1)). \end{aligned} \tag{7.3}$$

Use Fatou's Lemma, (7.2) and (7.3), while taking means in (7.1), to see that

$$\begin{aligned} \mathbb{P}\big(u^n_t(k)\big) &\le \langle u_0, p_t(\cdot,k)\rangle + 2c(t,k)e^{-\overline{\lambda}n} e^{\lambda'|k|} + tc(t,\overline{\lambda},\lambda') e^{-\overline{\lambda}n} e^{\lambda'|k|} \\ &\le P_t u_0(k) + c(t,\overline{\lambda},\lambda') e^{-\overline{\lambda}n} e^{\lambda'|k|} \quad \text{for any } \lambda' > 0, \text{ and } \overline{\lambda} > 0 \\ &\qquad\qquad \text{sufficiently small.} \end{aligned} \tag{7.4}$$

First bound $\langle u^n_{t'_N}, \phi_{t'_N}\rangle$ by the right side of (7.1) without the $\int_0^{t'_N}$-$\alpha(r)\,dr$ term, use (7.2) and (7.3), and then use the analogous bounds for v^n and Fatou's Lemma to conclude that for $\lambda' > 0$, and $\overline{\lambda} > 0$ (sufficiently small, depending on λ'),

$$\begin{aligned} \mathbb{P}\big(u^n_t(k) v^n_t(k')\big) &\le \varliminf_{N\to\infty} \mathbb{P}\left(u^n_{t'_N}\left(p_{t-t'_N}(\cdot,k)\right) v^n_{t'_N}\left(p_{t-t'_N}(\cdot,k')\right)\right) \\ &\le \left(\langle u_0, p_t(\cdot,k)\rangle + 2c(t,\overline{\lambda})e^{-\overline{\lambda}n} e^{\lambda'|k|} + tc(t,\overline{\lambda},\lambda') e^{-\lambda n} e^{\lambda'|k|}\right) \\ &\quad \cdot \left(\langle v_0, p_t(\cdot,k')\rangle + 2\tilde{c}(t,\overline{\lambda})e^{-\overline{\lambda}n} e^{\lambda'|k'|} + t\tilde{c}(t,\overline{\lambda},\lambda') e^{-\overline{\lambda}n} e^{\lambda'|k'|}\right) \\ &\le P_t u_0(k) P_t v_0(k') + c(t,\overline{\lambda},\lambda') e^{-\overline{\lambda}n} e^{\lambda'(|k|+|k'|)} \quad \text{(by } (H_2)). \end{aligned} \tag{7.5}$$

Use (7.4) and (7.5) in $(LS)_n$ to see that for $\lambda, T > 0$, if $\lambda' > 0$ is chosen to be sufficiently small, then for $\overline{\lambda} = \overline{\lambda}(\lambda')$ sufficiently small,

$$\begin{aligned}
\mathbb{P}\big(\sup_{t\le T}\langle u_t^n, \phi_{-\lambda}\rangle\big) &\le \langle u_0, \phi_{-\lambda}\rangle + \sum_{k\in S_n}\sum_j \int_0^T \mathbb{P}\big(u_s^n(j)\big)|q_{jk}|e^{-\lambda|k|}\,ds \\
&\quad + c\mathbb{P}\left[\left(\int_0^T \sum_{k\in S_n} e^{-2\lambda|k|}\gamma u_s^n(k)v_s^n(k)\,ds\right)^{1/2}\right] \\
&\le \langle u_0, \phi_{-\lambda}\rangle + \sum_k\sum_j \int_0^T \left(P_s u_0(j) + c(T,\overline{\lambda},\lambda')e^{-\overline{\lambda}n}e^{\lambda'|j|}\right)|q_{jk}|e^{-\lambda|k|}\,ds \\
&\quad + c\left[\int_0^T \sum_k e^{-2\lambda|k|}\gamma\left(P_s u_0(k)P_s v_0(k) + c(T,\overline{\lambda},\lambda')e^{-\overline{\lambda}n+\lambda'|k|2}\right)ds\right]^{1/2} \\
&\le \langle u_0, \phi_{-\lambda}\rangle + c(T,\lambda) < \infty,
\end{aligned}$$

where we have used the choice of λ' and (H_2) in the last line. Hence by symmetry we have

$$\sup_n \mathbb{P}\left(\sup_{t\le T}\langle u_t^n + v_t^n, \phi_{-\lambda}\rangle\right) < \infty \quad \forall\lambda, T > 0. \tag{7.6}$$

Now let $\Omega = \big(C(\mathbb{R}_+,\mathbb{R}_+)^2 \times C(\mathbb{R}_+,\mathbb{R})^2\big)^{\mathbb{Z}^d}$ with its Borel σ-field $\mathcal{F}$ and canonical right-continuous filtration $(\mathcal{F}_t)$. Standard weak convergence arguments and (7.6) will now show that $\{\big(u_\cdot^n(k), v_\cdot^n(k), B_\cdot^k, W_\cdot^k\big) : k \in \mathbb{Z}^d\}$ is a tight sequence in Ω and any weak limit point $\{\big(u_\cdot(k), v_\cdot(k), B^k, W^k\big) : k \in \mathbb{Z}^d\}$ (with law $\mathbb{P}$) will be a solution of $(LS)_{u_0,v_0}$ on $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$. For example, to show tightness of the stochastic integrals in the definition of $u_t^n(k)$ (in $(LS)_n$) one may work with the predictable square functions, use exponential bounds on the increments of a continuous martingale in terms of the increments of the square function (use Dubins-Schwarz) and utilize the fact that $\sup_{s\le t} u_s^n(k)v_s^n(k)$ is bounded in probability uniformly in n for each $t > 0$ and k in $\mathbb{Z}^d$ (by (7.6)). The tightness of the drifts in the definition of $u_t^n(k)$ is clear from (7.6). (7.6) also shows that $u_t(\cdot)$ and $v_t(\cdot)$ are M_{tem}-valued. The continuity in M_{tem} will follow from (b)(i) below.

Another approach to the construction of solutions to (LS) is to use the finite dimensional approach of Shiga and Shimizu (1980) (see the proof of their Theorem 2.1). This again uses (7.6) and (7.5).

(b)(i) Let (u, v) be a solution to $(LS)_{u_0,v_0}$ where $u_0, v_0 \in M_{tem}$. Define u_t^n by

$$u_t^n(k) = u_0(k) + \int_0^t \sum_{j\in S_n} u_s(j)q_{jk}\,ds + \int_0^t \big(\gamma u_s(k)v_s(k)\big)^{1/2}\,dB_s^k, \quad t \ge 0, k \in \mathbb{Z}^d.$$

Note that $\lim_{n\to\infty} u_t^n(k) = u_t(k)$. If $\phi_s^n(j) = 1_{S_n}(j)p_{t-s}(j,k)$ for $s \le t$ (t, k fixed) then a simple calculation (similar to, but simpler than, that of (7.1)) shows that for $s \le t$,
(7.7)

$$\langle u_s^n, \phi_s^n\rangle = \langle u_0, \phi_0^n\rangle + N_s^{n,u}(\phi) - \int_0^s \sum_{j\in S_n}\sum_{j'\notin S_n} u_r(j)q_{jj'}p_{t-r}(j',k)\,dr \le \langle u_0,\phi_0^n\rangle + N_s^{n,u}(\phi),$$

$$N_s^{n,u}(\phi) = \sum_{j\in S_n}\int_0^s \big(\gamma u_r(j)v_s(j)\big)^{1/2} p_{t-r}(j,k)\,dB_r^j.$$

As in (a), Fatou's Lemma and a stopping argument leads to

$$\mathbb{P}\big(u_t(k)\big) \le \varliminf_{n\to\infty} \mathbb{P}\big(u_t^n(k)\big) \le P_t u_0(k), \quad \text{and similarly} \quad \mathbb{P}\big(v_t(k)\big) \le P_t v_0(k), \tag{7.8}$$

and

$$\mathbb{P}\big(u_t(k)v_t(k')\big) \le \varliminf_{n\to\infty} \mathbb{P}\big(u_t^n(k)v_t^n(k)\big) \le P_t u_0(k) P_t v_0(k). \tag{7.9}$$

Argue as in the derivation of (7.6) (now using (7.8) and (7.9)) to see that for any $\lambda, T > 0$

$$\mathbb{P}\left(\sup_{t\le T}\langle u_t + v_t, \phi_{-\lambda}\rangle\right) = \lim_{n\to\infty}\mathbb{P}\left(\sup_{t\le T}\langle u_t+v_t, 1_{S_n}\phi_{-\lambda}\rangle\right) < \infty.$$

The same reasoning shows that

$$\lim_{n\to\infty}\mathbb{P}\left(\sup_{t\le T}\langle u_t+v_t, 1_{S_n^c}\phi_{-\lambda}\rangle\right) = 0 \quad \forall \lambda, T>0.$$

It follows that $\langle u_t, \phi_{-\lambda}\rangle$ is a.s. continuous in t since it is the uniform limit of $\langle u_t, 1_{S_n}\phi_{-\lambda}\rangle$, $t\in[0,T]$ as $n\to\infty$ a.s. This shows that $(u_\cdot, v_\cdot) \in \Omega_{tem}$ a.s.

(ii) Let $\phi \in M_{rap}^s$, fix $t > 0$ and let $\phi_n(s,k) = I_{S_n}(k)P_{t-s}\phi(k)$ for $s \le t$. If $N_s^u(\phi_n)$ is defined as $N_s^u(t,\phi)$ in the statement of the Theorem 2.2(b)(ii), but with k summed over S_n, then Itô's Lemma gives

$$\langle u_t, \phi_n(t)\rangle = \langle u_0, \phi_n(0)\rangle + \int_0^t \langle u_s, \dot\phi_n(s)\rangle + \langle u_s Q, \phi_n(s)\rangle\,ds + N_t^u(\phi_n). \tag{7.10}$$

Note that

$$\begin{aligned}
&\mathbb{P}\left(\int_0^t |\langle u_s, \dot\phi_n(s)\rangle + \langle u_sQ, \phi_n(s)\rangle|\,ds\right)\\
&= \mathbb{P}\left(\int_0^t \left| -\sum_{j\in S_n}\sum_{k\in S_n^c} u_s(j)q_{jk}P_{t-s}\phi(k) + \sum_{j\in S_n^c}\sum_{k\in S_n} u_s(j)q_{jk}P_{t-s}\phi(k)\right|\,ds\right)\\
&\to 0 \quad \text{as} \quad n\to\infty,
\end{aligned}$$

where in the last line we have used (7.8), (H_2), and argued as in the derivation of (7.3). (7.9) shows that $N_t^u(\phi_n)$ is a square integrable martingale and also shows that for $n > m$, and $\lambda' > 0$

$$\begin{aligned}
\mathbb{P}\big(\langle N^u(\phi_n) - N^u(\phi_m)\rangle_t\big) &\le \mathbb{P}\left(\int_0^t \sum_{k\in S_n - S_m} \gamma P_{t-s}\phi(k)^2 P_s u_0(k) P_s v_0(k)\, ds\right) \\
&\le c(t,\lambda') \int_0^t \sum_{k\in S_n - S_m} P_{t-s}\phi(k)^2 e^{2\lambda'|k|}\, ds \\
&\le c(t,\lambda')\|\phi\|_\infty \int_0^t \sum_{k\in S_n - S_m} P_{t-s}\phi(k) e^{2\lambda'|k|}\, ds \\
&\to 0 \quad \text{as} \quad n,m\to\infty \quad \text{by } (H_2).
\end{aligned}$$

The above bounds allow us to prove (ii) by letting $n\to\infty$ in (7.10), using a similar argument for $\langle v_t, \psi\rangle$, and noting that the orthogonality of the martingales is obvious.

(iii) is immediate from (ii) and a monotone class argument.

(iv) For ϕ as in the statement of (iv), if $\phi_n(j) = I_{S_n}(j)\phi(j)$ we may write

$$\langle u_t, \phi_n\rangle = \langle u_0, \phi_n\rangle + \int_0^t \langle u_s Q, \phi_n\rangle\, ds + \sum_{k\in S_n} \phi(k) \int_0^t \sqrt{\gamma u_s v_s(k)}\, dB_s^k.$$

One now may easily prove (iv) by letting $n\to\infty$ and using the moments obtained in (iii) and (H_2) to see that the martingale term converges in L^2 and the drift term converges in L^1.

(c) Let $u_0, v_0 \in M_{rap}$. If $\lambda > 0$ we may use (b)(iv) with $\phi = 1_{S_n}\phi_\lambda$ to see that

$$\begin{aligned}
\mathbb{P}\left(\sup_{t\le T}\langle u_t, 1_{S_n^c}\phi_\lambda\rangle\right) &\le \langle u_0, 1_{S_n^c}\phi_\lambda\rangle + c\mathbb{P}\left(\left(\sum_{k\in S_n^c}\int_0^T \gamma u_s(k) v_s(k)\phi_\lambda(k)^2\, ds\right)^{1/2}\right) \\
&\quad + \int_0^T \sum_{k\in S_n^c}\sum_j \mathbb{P}\big(u_s(j)|q_{jk}|\phi_\lambda(k)\big)\, ds \\
&\le \langle u_0, 1_{S_n^c}\phi_\lambda\rangle + c \sum_{k\in S_n^c}\int_0^T \gamma P_s u_0(k) P_s v_0(k)\phi_\lambda(k)^2\, ds^{1/2} \\
&\quad + \int_0^T \sum_{k\in S_n^c}\sum_j P_s u_0(j)|q_{jk}|\phi_\lambda(k)\, ds \\
&\to 0 \quad \text{as} \quad n\to\infty
\end{aligned}$$

by an easy argument using (H_2). From this it is easy to see that $(u_t, v_t)\in\Omega_{rap}$ a.s. and $\mathbb{P}(\sup_{t\le T}\langle u_t + v_t, \phi_\lambda\rangle) < \infty$. It is also clear from the above that if $|\phi| + |\psi| \le c\phi_\lambda$ for some $c, \lambda > 0$, then we may apply (b)(iv) to $1_{S_n}\phi$ and $1_{S_n}\psi$ and let $n\to\infty$ to get (iv) for ϕ and ψ. The derivation of b(ii) for ϕ, ψ in M^s_{tem} is similar.

(d) Let $u_0, v_0 \in M_F$. Use the bounds

$$\sum_k P_s u_0(k) P_s v_0(k) = \sum_i \sum_j u_0(i) p_{2s}(i,j) v_0(j) \le \langle u_0, 1\rangle\langle v_0, 1\rangle < \infty$$

and

$$\begin{aligned}\sum_k \sum_j P_s u_0(j)|q_{jk}| &\le 2\|q\|_\infty \sum_j P_s u_0(j) \quad \text{(recall } \sum_k |q_{jk}| = 2|q_{jj}|) \\ &= 2\|q\|_\infty \langle u_0, 1\rangle < \infty,\end{aligned}$$

and argue as in (c) with $\lambda = 0$ to see that $\mathbb{P}(\sup_{t\le T}\langle u_t, 1\rangle) < \infty$, $(u_\cdot, v_\cdot) \in \Omega_F$ a.s., and b(iv) holds for $\phi, \psi \in M^s_b$. The derivation of (b)(ii) for $\phi, \psi \in M^s_b$ is now also clear. Since $Q1 = 0$, the last assertion is immediate from the above extension of b(iv) with $\phi = \psi \equiv 1$. ∎

We next turn to the proof of Theorem 6.1, the corresponding result for the stochastic p.d.e. We need to work with super-Brownian motions with initial condition $u_0(x)\,dx$ where $u_0 \in C^+_{tem}$ and as we don't know of a proper reference, a terse outline of the theory is presented. If $\sigma : \mathbb{R} \to \mathbb{R}_+$ is bounded and measurable, $s_0 \ge 0$, $u_0 \in C^+_{tem}$ and $\dot W$ is a white noise on $\mathbb{R}_+ \times \mathbb{R}$, Theorem 2.5 of Shiga (1994) establishes the existence of a continuous C^+_{tem}-valued solution, u, of

$$(SP) \qquad \begin{aligned}&\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t,x) + \big(\sigma(x)u(t,x)\big)^{1/2}\dot W, \quad t \ge s_0 \\ &u(s_0, x) = u_0(x).\end{aligned}$$

(The methods in Shiga's paper easily accommodate a variable σ.) A precise formulation of a solution to (SP) on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is given by the obvious analogue of that for $(SPDE)_{u_0,v_0}$. If μ is a measure and ϕ is a real-valued function $\mu(\phi)$ denotes the integral $\int \phi\, d\mu$. We now set $s_0 = 0$ for convenience. The measure-valued process $X_t(dx) = u(t,x)\,dx$ clearly satisfies the martingale problem

$$(MP)_{u_0,\sigma} \quad \begin{aligned}&\forall\phi \in C^2_{rap},\ M_t(\phi) = X_t(\phi) - \langle u_0, \phi\rangle - \int_0^t X_s(\phi''/2)\,ds \quad \text{is a continuous} \\ &(\mathcal{F}_t) - \text{local martingale with} \quad \langle M(\phi)\rangle_t = \int_0^t X_s(\sigma^2\phi^2)\,ds.\end{aligned}$$

We need only check the extension from C^2_c to C^2_{rap}. For this, suppose more generally that $\phi \in C^2_\lambda$ for some $\lambda > 0$ and let h_N be a smooth function with support in $[-N, N]$ which equals one on $[-N+1, N-1]$. A simple application of the fundamental theorem of calculus shows that $|\phi'|_\lambda \le \lambda^{-1}|\phi''|_\lambda < \infty$ and this allows us to use $(MP)_{u_0,\sigma}$ with $\phi_N = h_N\phi$ in place of ϕ and let $N \to \infty$ to derive $(MP)_{u_0,\sigma}$ for ϕ.

Let M_{tem} denote the space of measures, μ, on $\mathbb{R}$ such that $\mu(\phi_{-\lambda}) < \infty\ \forall\lambda > 0$, and topologize M_{tem} so that $\mu_n \to \mu$ in M_{tem} iff $\lim_{n\to\infty} \mu_n(\phi) = \mu(\phi)\ \forall\phi \in C_c(\mathbb{R}) \cup \{\phi_{-\lambda} : \lambda > 0\}$. One readily defines a metric ρ on M_{tem} so that (M_{tem}, ρ) is Polish. In $(MP)_{u_0,\sigma}$ we will implicitly assume that $X.$ is an $(\mathcal{F}_t)$-adapted continuous M_{tem}-valued process. As in Chapter 2 of Walsh (1986), the martingale measure M in $(MP)_{u_0,\sigma}$ extends to integrands which are $\mathcal{P}(\mathcal{F}.) \times \mathcal{B}$-measurable ($\mathcal{P}(\mathcal{F}.)$ is the predictable σ-field on $\mathbb{R}_+ \times \Omega$ and $\mathcal{B}$ is the Borel σ-field on $\mathbb{R}$) and satisfy $\int_0^T X_s(\sigma^2\phi_s^2)\, ds < \infty\ \forall T > 0$ a.s. Then $M_t(\phi) = \int_0^t \int \phi(s,x,\omega)\, dM(s,x)$ is still a continuous $(\mathcal{F}_t)$-local martingale with square function $\int_0^t X_s(\sigma^2\phi_s^2)\, ds$. $(MP)_{u_0,\sigma}$ extends in the obvious way to those functions $\phi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ such that $t \to \phi(t,\cdot)$ and $t \to (\frac{\partial\phi}{\partial t} + \frac{1}{2}\frac{\partial^2\phi}{\partial x^2})(t,\cdot)$ are continuous C_λ-valued functions on $[0,T]$, where $\lambda > 0$ is fixed. For example, one may use Proposition 1.3.3 of Ethier-Kurtz (1986) and the generator of space-time Brownian motion to bootstrap up from $\{\phi(s,x) = \phi_1(s)\phi_2(x) : \phi_1 \in C^1([0,T]), \phi_2 \in C^2_\lambda\}$. If $\phi \in C^{2,+}_\lambda$, let $U_t\phi(x)$ denote the unique solution of

$$\frac{\partial U_s}{\partial s} = \frac{1}{2}\frac{\partial^2}{\partial x^2}U_s - \frac{\sigma^2}{2}U_s^2, \quad U_0 = \phi.$$

P_t is a strongly continuous semigroup on C_λ (Lemma 6.2 of Shiga (1994)) and so $t \to U_t\phi$ is a continuous C^+_λ-valued map. Let $\phi(s,x) = U_{T-s}\phi(x)$ in (the extended) $(MP)_{u_0,\sigma}$ (valid because $\phi_s + \frac{1}{2}\phi_{xx} = \sigma^2\phi_s^2/2$) and use Itô's lemma as in the usual uniqueness proof for superprocesses to see that

$$\mathbb{P}\left(\exp\{-X_T(\phi)\}\right) = \exp\bigl(-\langle u_0, U_T\phi\rangle\bigr).$$

This gives the uniqueness in law of solutions to $(MP)_{u_0,\sigma}$ and therefore the uniqueness in law (on $C(\mathbb{R}_+, C^+_{tem})$) of solutions to (SP). Let $\mathbb{Q}_{s_0,u_0,\sigma}$ denote the law on $C(\mathbb{R}_+, C^+_{tem})$ of the unique (in law) solution of (SP).

Proof of Theorem 6.1. (a) Let $(\Omega, \mathcal{F}, \mathcal{F}_t)$ be Ω_{tem} with its Borel σ-field and canonical right-continuous filtration. The coordinate variables on Ω_{tem} will be denoted by (u_t, v_t) and $\mathcal{F}[s,t] = \sigma\bigl((u_r, v_r) : s \le r \le t\bigr)$. Let $u^{(n)}(t,x) = u\bigl([tn]/n, x\bigr) \wedge n$ and similarly define $v^{(n)}(t,x)$. $\mathbb{Q}^n$ is the unique law on $(\Omega, \mathcal{F})$ such that for all i in $\mathbb{Z}_+$ and A in $\mathcal{F}[i/n, (i+1)/n]$,

$$\mathbb{Q}^n\left((u,v) \in A \mid \mathcal{F}_{i/n}\right) = \mathbb{Q}_{i/n, u_{i/n}, \gamma v^{(n)}_{i/n}} \times \mathbb{Q}_{i/n, v_{i/n}, \gamma u^{(n)}_{i/n}}(A).$$

Hence given $\mathcal{F}_{i/n}$, on $[i/n, (i+1)/n]$, $(u., v.)$ evolves like a pair of independent super-Brownian motions with branching rates $\gamma v^{(n)}_{i/n}$ and $\gamma u^{(n)}_{i/n}$, respectively. It follows easily from $(MP)_{u_0,\sigma}$ that

$$(ME) \qquad \mathbb{Q}^n\left(\langle u_t, \phi\rangle\right) = \langle u_0, P_t\phi\rangle, \quad \mathbb{Q}^n\left(\langle v_t, \psi\rangle\right) = \langle v_0, P_t\psi\rangle \quad \text{for} \quad \phi, \psi \in C_{rap},$$

and that if $\phi,\ \psi \in C^2_{rap}$, then

$$M^1_t(\phi) = \langle u_t, \phi\rangle - \langle u_0, \phi\rangle - \int\limits_0^t \langle u_s, \phi''/2\rangle\, ds \quad \text{and}$$

$$M^2_t(\psi) = \langle v_t, \psi\rangle - \langle v_0, \psi\rangle - \int\limits_0^t \langle v_s, \psi''/2\rangle\, ds$$

are orthogonal continuous $(\mathcal{F}_t)$-local martingales such that

$$\langle M^1(\phi)\rangle_t = \int\limits_0^t \langle \gamma v_s^{(n)} u_s, \phi^2\rangle\, ds \quad \text{and} \quad \langle M^2(\psi)\rangle_t = \int\limits_0^t \langle \gamma u_s^{(n)} v_s, \psi^2\rangle\, ds.$$

Denote the above martingale problem by $(MP)_n$. As for superprocesses, $M^1_t(\phi)$ and $M^2_t(\psi)$ extend to orthogonal martingale measures and to the usual class of predictable integrands, and $(MP)_n$ extends to time dependent coefficients including $\phi(s,y) = p_{t+\epsilon-s}(y-x) \equiv p_{t+\epsilon-s}(y,x)$, $s \le t$, for $\epsilon > 0$. This leads to

$$\langle u_t, p_\epsilon(\cdot, x)\rangle = P_{t+\epsilon} u_0(x) + \int\limits_0^t \int p_{\epsilon+t-s}(y,x) dM^1(s,y) \quad \mathbb{Q}^n\text{-a.s.,} \tag{7.11}$$

and Fatou's Lemma shows that

$$\mathbb{Q}^n\big(u(t,x)\big) \le P_t u_0(x), \mathbb{Q}^n\big(v(t,x)\big) \le P_t v_0(x), \mathbb{Q}^n\big(u(s,x)v(t,y)\big) \le P_s u_0(x) P_t v_0(y). \tag{7.12}$$

To check that we may let $\epsilon \downarrow 0$ in the stochastic integral in (7.11), note that for $\lambda > 0$

$$\begin{aligned}
&\mathbb{Q}^n\left(\int\limits_0^t \int \big(p_{\epsilon+t-s}(y,x) - p_{t-s}(y,x)\big)^2 v_s^{(n)}(y) u_s(y)\, dy\, ds\right) \\
&\le \int\limits_0^t \int \big(p_{\epsilon+t-s}(y,x) - p_{t-s}(y,x)\big)^2 P_{[sn]/n} v_0(y) P_s u_0(y)\, dy\, ds \quad \text{(by (7.12))} \\
&\le \int\limits_0^t \int \big(p_{\epsilon+t-s}(y,x) - p_{t-s}(y,x)\big)^2\, dy\, ds^{1/2} c(t,\lambda) \left(\int\limits_0^t \int \big(p_{\epsilon+t-s}(y,x)^2 + p_{t-s}(y,x)^2\big) e^{\lambda|y|}\, dy\, ds\right)^{1/2} \\
&\qquad\qquad \text{(Cauchy-Schwarz and Lemma 6.2(ii) of Shiga (1994))} \\
&\le c(t,\lambda)\epsilon^{1/4} \left(\int\limits_0^t (\epsilon+t-s)^{-1/2} + (t-s)^{-1/2}\, ds\right)^{1/2} e^{\lambda|x|/2} \quad \text{(Lemma 6.2 of Shiga (1994))} \\
&\le c(t,\lambda)\epsilon^{1/4} e^{\lambda|x|/2}.
\end{aligned}$$

Therefore, letting $\epsilon \downarrow 0$ in (7.11) gives

$$(7.13) \qquad u(t,x) = P_t u_0(x) + \int_0^t \int p_{t-s}(y,x) dM^1(s,y) \quad \mathbb{Q}^n\text{-a.s.}$$

The Gronwall argument in Shiga (1994) (Section 6) (which relies on (7.13) and its counterpart for $v(t,x)$) is now readily modified to prove

$$(7.14) \qquad \sup_n \sup_{s\le t} \int e^{-\lambda|x|} \mathbb{Q}^n \big(u(s,x)^q + v(s,x)^q\big)\, dx < \infty \quad \forall \lambda, q, t > 0.$$

Now proceed as in Section 6 of Shiga (1994) to see that $\{\mathbb{Q}^n\}$ is tight on Ω_{tem} (the truncation in the definition of $u^{(n)}$ and $v^{(n)}$ poses no difficulties). Let $\mathbb{P}$ be any limit point of $\{\mathbb{Q}^n\}$. Standard arguments, using $(MP)_n$ and (7.14) show that for ϕ, $\psi \in C^2_{rap}$, $M^1_t(\phi)$ and $M^2_t(\psi)$ (defined in $(MP)_n$) are square integrable continuous $(\mathcal{F}_t)$-martingales under $\mathbb{P}$ such that

$$\langle M^1(\phi)\rangle_t = \int_0^t \langle \gamma u_s v_s, \phi^2\rangle\, ds, \quad \langle M^2(\psi)\rangle_t = \int_0^t \langle \gamma u_s v_s, \psi^2\rangle\, ds, \quad \text{and} \quad \langle M^1(\phi), M^2(\psi)\rangle_t = 0.$$

By enlarging the probability space to include independent white noises $\overline{W}_1$ and $\overline{W}_2$, one may easily define mutually independent white noises (W_1, W_2) in terms of $(M_1, M_2, \overline{W}_1, \overline{W}_2)$ so that (u, v, W_1, W_2) solves $(SPDE)_{u_0,v_0}$. This proves (a).

(b) Let (u, v, W_1, W_2) be a solution to $(SPDE)_{u_0,v_0}$ on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. The representation in (iv) for ϕ, $\psi \in C^2_{rap}$ (or even C^2_λ for some $\lambda > 0$) follows from the corresponding result for ϕ, ψ in C_c by approximating by $h_N\phi$ and $h_N\psi$, as for $(MP)_{u_0,\sigma}$. Note that the fact that u_t and v_t are C^+_{tem}-valued is used here. The integrability assertions in (iv) will be immediate from (iii). As in our discussion for superprocesses we may extend the semimartingale decomposition in (iv) in the obvious manner to $\phi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ such that $s \to \phi(s,\cdot)$ and $s \to \frac{\partial \phi}{\partial s}(s,\cdot) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\phi(s,\cdot)$ are continuous C_λ-valued maps on $[0,t]$ for fixed λ, $t > 0$. If $\phi \in C^2_{rap}$, these conditions are satisfied by $\phi(s,x) = P_{t-s}\phi(x)$ and lead to the representation in (i). If $\phi \in C_{rap}$, then we have the representation in (i) for $P_\epsilon \phi \in C^2_{rap}$ and by taking $\epsilon \downarrow 0$ we get this representation for ϕ (the square functions of the martingale term converge in probability because $\sup_{s\le t} |P_s\phi|_\lambda < \infty$ and $(u_\cdot, v_\cdot) \in \Omega_{tem}$ a.s.)

To prove (ii), fix $\epsilon > 0$, $x \in \mathbb{R}$ and use (i) with $\phi(y) = p_\epsilon(x-y)$. Now let $\epsilon \downarrow 0$ and use Fatou's lemma (both with respect to ϵ and an appropriate family of stopping times to handle the stochastic integrals) to see that

$$\mathbb{P}\big(u(t,x)\big) \le P_t u_0(x), \quad \mathbb{P}\big(v(t,x)\big) \le P_t v_0(x), \quad \mathbb{P}\big(u(t,x)v(t,x')\big) \le P_t u_0(x) P_t v_0(x').$$

The representations in (ii) are now obtained by letting $\epsilon \downarrow 0$ in the above representation, as in the derivation of (7.13) and the square integrability of the stochastic integrals is also immediate from this argument. The first three equalities in (iii) are immediate from (ii) and the last three follow by Fubini's theorem. Finally (6.1) may be proved by a Gronwall argument using (ii) as in Section 6 of Shiga (1994).

(c) Let (u, v) be as above but with $u_0, v_0 \in C^+_{rap}$. Let ϕ, $\psi \in C^2_{tem}$, h_N be as in the derivation of $(MP)_{u_0,\sigma}$ and set $(\phi_N, \psi_N) = (h_N\phi, h_N\psi)$. Apply (b)(iv) to (ϕ_N, ψ_N) and let $N \to \infty$ to see that it remains valid for (ϕ, ψ). The first set of moments in (b)(iii) show that the stochastic integral terms converge in L^2 uniformly in $t \leq T$ and the drifts converge in total variation in L^1. Note that in fact (iv) holds if ϕ, $\psi \in C^2_\lambda$ for some $\lambda > 0$. As in (b), but with $C_{-\lambda}$ in place of C_λ, b(i) now follows for ϕ in C_{tem}.

To prove (6.2) we modify the proof of Theorem 2.4 in Shiga (1994). By the above argument we see that $t \to \int(\cosh \lambda x)u(t, x)\, dx$ is continuous for all $\lambda \in \mathbb{R}$. Since $(u., v.) \in \Omega_{tem}$, this shows that for T, λ, $q > 0$

$$\begin{aligned}\sup_{t\leq T} \int \phi_\lambda(x) u(t, x)^q\, dx &\leq c(T, q, \omega) \sup_{t\leq T} \int e^{(\lambda+q-1)|x|} u(t, x)\, dx \\ &\leq 2c(T, q, \omega) \sup_{t\leq T} \int \cosh\,\big((\lambda + q - 1)x\big) u(t, x)\, dx < \infty \quad \text{a.s.}\end{aligned}$$

Therefore

$$T_k = \inf\left\{t : \int e^{\lambda|x|}\big(u(t, x)^q + v(t, x)^q\big)\, dx > k\right\} \uparrow \infty \quad \text{a.s.} \quad \text{as} \quad k \to \infty.$$

Shiga's Gronwall argument is now easily modified to show that

$$\sup_k \sup_{r\leq T} \mathbb{P}\left(\int e^{\lambda|x|}\big(u(r, x)^q + v(r, x)^q\big)\, dx\ 1(r \leq T_k)\right) < \infty \quad \forall q, \lambda, T > 0.$$

Fatou's Lemma now gives (6.2). Arguing as in the proof of Theorem 2.4 of Shiga (1994) (using a suitably modified version of Lemma 6.3(iii) of that paper), one easily shows that $(u., v.) \in \Omega_{rap}$ a.s.

(d) If $u_0, v_0 \in C^+_{int}$, the validity of (b)(i) and (iv) for ϕ, ψ as in the statement of the theorem follows as in (c) (use C_b in place of C_λ). Set $\phi = \psi = 1$ to get the last assertion which implies $(u_t, v_t) \in (C^+_{int})^2$ $\forall t \geq 0$ a.s. ∎

Acknowledgement. We thank Carl Mueller for suggesting the mutually catalytic spde to us and for several stimulating discussions on this topic.

REFERENCES

Dawson, D.A. (1977). The critical measure diffusion process. **Z. Wahrsch. verw. Gebiete 40**, 125–145.

Dawson, D.A. and Fleischmann, K. (1988). Strong clumping of critical space-time branching models in subcritical dimensions. **Stoch. Proc. Appl. 30**, 193–208.

Dawson, D.A. and Fleischmann, K. (1994). A super-Brownian motion with a single point catalyst. **Stoch. Proc. Appl. 49**, 3–40.

Dawson, D.A. and Fleischmann, K. (1995). Super-Brownian motions in higher dimensions with absolutely continuous measure states. **Journ. Theoret. Probab. 8**, 179–206.

Dawson, D.A. and Hochberg, K. (1979). The carrying dimension of a stochastic measure diffusion. **Ann. Probab. 7**, 693–703.

Dawson, D.A. and March, P. (1995). Resolvent estimates for Fleming-Viot operators and uniqueness of solutions to related martingale problems. **Journ. Funct. Anal. 132**, 417-472.

Dellacherie, C. and Meyer, P.A. (1978). **Probabilities and Potential**. North Holland Mathematical Studies 29, North Holland, Amsterdam.

Delmas, J-F. (1996). Super-mouvement Brownien avec catalyse. **Stochastics and Stochastic Reports 58**, 303–347.

Ethier, S.N. and Kurtz, T.G. (1986). **Markov Processes: Characterization and Convergence**, Wiley, New York.

Gauthier, G. (1996). Multilevel bilinear systems of stochastic differential equations. preprint.

Ikeda, N. and Watanabe, S. (1981). **Stochastic Differential Equations and Diffusion Processes**. North Holland, Amsterdam.

Lawler, G.F. (1991). **Intersections of Random Walks**. Birkhauser, Boston.

Liggett, T.M. (1985). **Interacting Particle Systems**. Springer-Verlag, New York.

Mueller, C. and Perkins, E.A. (1997). Extinction for some parabolic stochastic p.d.e.'s on the lattice. In preparation.

Mytnik, L. (1996). **Superprocesses in Random Environments and Related Topics**, Ph.D. Thesis, Technion.

Perkins, E.A. (1988). A space-time property of a class of measure-valued branching diffusions. **Trans. Amer. Math. Soc. 305**, 743–795.

Perkins, E.A. (1995). **On the Martingale Problem for Interactive Measure-valued Branching Diffusions**. Memoirs of the A.M.S., 115, no. 549.

Shiga, T. (1994). Two contrasting properties of solutions for one-dimensional stochastic partial differential equations. **Can. J. Math. 46**, 415–437.

Shiga, T. and Shimizu, A. (1980). Infinite dimensional stochastic differential equations and their applications. **J. Math. Kyoto U. 20**, 395–416.

Stroock, D.W. and Varadhan, S.R.S (1979). **Multidimensional diffusion processes**. Springer-Verlag, Berlin.

Walsh, J.B. (1986). An Introduction to Stochastic Partial Differential Equations, in **Ecole d'été**

de Probabilités Saint-Flour XIV 1984, P.L. Hennequin (ed.), Lect. Notes Math. 1180, 265–439.

Williams, D. (1991). **Probability with Martingales**. Cambridge U. Press, Cambridge.

Yor, M. (1978). Continuité des temps Locaux, in Temps Locaux, **Astérisque 52–53**, 17–22.

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