# THE MULTIFRACTAL STRUCTURE OF SUPER-BROWNIAN MOTION

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**Summary**

We calculate the multifractal spectrum and mass exponents for super-Brownian motion in three or more dimensions. The former is trivial for points of unusually high density but not for points in the support of unusually low density. This difference is due to the presence of sets of points in the support (of positive dimension) about which there are asymptotically large empty annuli. This behaviour is quite different from that of ordinary Brownian motion and invalidates the multifractal formalism in the physics literature. The mass exponents for packing and Hausdorff measure are distinct, and both are piecewise linear.

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## 1. Introduction.

Formal methods for analyzing a finite Borel measure $\mu$ on $\mathbb{R}^d$ through its "multifractal spectrum" have been proposed by several authors—see Halsey et al (1986) [Ha]. These methods deal with a mass exponent $b(q)$ and a multifractal spectrum function $f(\alpha)$. Formally

$$b(q) = \lim_{n\to\infty} \log\left(\sum_C \mu(C)^q\right)(n\log 2)^{-1}$$

where the sum is over cubes $C$ of edge length $2^{-n}$ such that the vertices of $2^n C$ are in $\mathbb{Z}^d$, (i.e. dyadic cubes) and

$$f(\alpha) = \dim\{x \in S(\mu) : d(\mu, x) \equiv \lim_{r\downarrow 0} \frac{\log \mu\big(B(x,r)\big)}{\log r} = \alpha\}.$$

Here $S(\mu)$ denotes the closed support of $\mu$ and $\dim A$ is the Hausdorff dimension of $A$. It is not hard to see heuristically that $f$ is the Legendre transform of $b$,

$$f(\alpha) = \inf_q(\alpha q + b(q)). \tag{1.1}$$

There are a host of difficulties presented by the above formalisms (such as the existence of the above limits). A recent paper of Olsen (1995) [O] gives an exhaustive analysis of the multifractal formalism.

In all cases where results such as (1.1) have been rigorously proved, $\mu$ satisfies extremely strong uniform regularity conditions. In this work we will study the multifractal structure of $\mu = X_t$ where $X$ is super-Brownian motion on $\mathbb{R}^d$. Basic facts about super-Brownian motion are recalled in Section 3. For now one should know that it is a continuous Markov process taking values in the space $M_F(\mathbb{R}^d)$ of finite measures on $\mathbb{R}^d$ and it is a.s. singular iff $d \geq 2$. $\mathbb{P}_{m_0}$ denotes the law of this process on $C\left(\mathbb{R}_+, M_F(\mathbb{R}^d)\right) = \Omega_X$, starting from $m_0$. We avoid the critical two-dimensional case and consider only $d \geq 3$ throughout this work. The regularity conditions mentioned above will not hold and we shall see that the behaviour of $f(\alpha)$ is quite different depending on whether or not one replaces the limit in its definition by a lim sup or lim inf.

The fractal properties of $S(X_t)$ are known. If $\phi - m$ denotes Hausdorff measure with respect to $\phi(s) = s^2 \log\log 1/s$ then (Perkins (1989), Dawson and Perkins (1991) [DP]) there is a constant $c_d > 0$ such that

$$X_t(A) = c_d\phi - m\big(A \cap S(X_t)\big) \quad \forall \text{ Borel set } A \quad \mathbb{P}_{m_0} - \text{a.s.} \quad \forall\, t > 0. \tag{1.2}$$

More recently LeGall, Perkins and Taylor (1995) [LPT] showed there is no exact packing function $\psi(s)$ satisfying the analogue of (1.2) for packing measure $\psi - p$. Instead for every $\psi$, $\psi - p\,(S(X_t))$ is zero a.s. or infinite a.s. and these two possibilities are distinguished by an integral test for $\psi$ at the origin—see Theorem 1.2 of [LPT]. These packing results are different from the covering (Hausdorff measure) results because they depend on the lower tail of $X_t\big(B(x,r)\big)/r^2$ $\big(B(x,r) = \{y \in \mathbb{R}^d : |y-x| < r\}\big)$ for $x$ in $S(X_t)$ and $r$ small as opposed to the upper tail which enters for the covering results. The former is "fat" (see Lemma 3.2 and Proposition 4.5 of [LPT]) while the latter is a negative exponential. This difference led us to look at the multifractal structure of $X_t$.

Theorem 5.5 of [DP] and Theorem 1.1 of [LPT] imply that

$$(1.3) \qquad d(X_t, x) = \lim_{r \downarrow 0} \frac{\log X_t\big(B(x,r)\big)}{\log r} = 2 \quad \text{for } X_t - \text{ a.a. } x \quad \mathbb{P}_{m_0} - \text{ a.s. } \forall\, t > 0.$$

Hence $X_t$ is dimension regular with exact dimension 2 in the sense of Cutler (1992); the local irregularities of $X_t$ do not affect $d(X_t, x)$ at a typical point. If $\mu$ is occupation measure up to time $t$ for the $d$-dimensional Brownian path $B$ then the results in Perkins and Taylor (1987) allow one to strengthen (1.3) to

$$(1.4) \qquad d(\mu, x) = 2 \quad \text{for all } x \text{ in } S(\mu) = \{B_s; s \le t\} \quad \text{a.s.}$$

We shall see that such a strengthening is false for $S(X_t)$. To examine the exceptional sets of points in $S(X_t)$ for which (1.3) may fail, we introduce (for $\mu$ in $M_F(\mathbb{R}^d)$)

$$\underline{d}(\mu, x) = \liminf_{r \downarrow 0} \frac{\log \mu\big(B(x,r)\big)}{\log r}; \ \overline{d}(\mu, x) = \limsup_{r \downarrow 0} \frac{\log \mu\big(B(x,r)\big)}{\log r}$$

and

$$A_\alpha(\mu) = \{x \in S(\mu) : d(\mu, x) = \alpha\}.$$

The smallness of the upper tail of $X_t\big(B(x,r)\big)/r^2$ will lead to (Theorem 4.1)

$$(1.5) \qquad \underline{d}(X_t, x) = 2 \quad \text{for all } x \text{ in } S(X_t) \quad \text{a.s.,}$$

that is, there are no points in $S(X_t)$ where the lim sup behaviour of $X_t\big(B(x,r)\big)$ as $r \downarrow 0$ differs (logarithmically) from that of $r^2$. (1.5) is established in Section 4 as a consequence of more precise uniform (in $x \in S(X_t)$) upper and lower bounds for the lim sup behaviour of $X_t\big(B(x,r)\big)$ as $r \downarrow 0$ (the former is due to Barlow, Evans and Perkins (1991) [BEP]). Note that (1.3) and (1.5) show that $A_\alpha(X_t) = \phi$ a.s. if $\alpha \neq 2$, and (for $\mu = X_t$) $f(\alpha) = 2 \times 1(\alpha = 2)$ a.s.

However, there are exceptional points $x$ in $S(X_t)$ for which $\overline{d}(X_t, x) > 2$. In fact we will show (Theorem 6.8)

(1.6)

$$\begin{aligned} \dim\{x \in S(X_t) : \overline{d}(X_t, x) = \alpha\} &= \dim\{x \in S(X_t) : \overline{d}(X_t, x) \ge \alpha\} \\ &= \frac{8}{\alpha} - 2 \text{ a.s. on } \{X_t \neq 0\} \quad \text{for } t > 0 \quad \text{and } 2 \le \alpha \le 4 \end{aligned}$$

and the above sets are empty if $\alpha > 4$. To study these sets of exceptionally thin points for $\mu$ in $M_F(\mathbb{R}^d)$, let

$$B_\alpha(\mu) = \{x \in S(\mu) : \overline{d}(\mu, x) = \alpha\}$$

and

$$C_\alpha(\mu) = \{x \in S(\mu) : \overline{d}(\mu, x) \geq \alpha\}.$$

In Section 5 we give a covering argument which gives the upper bound in (1.6) and shows $C_\alpha(X_t) = \phi$ a.s. for $\alpha > 4$ (Theorem 5.4). To prove the lower bound in (1.6) we introduce another closely related set of exceptional points.

**Definition 1.1.** If $\gamma \geq 1$ and $S \subset \mathbb{R}^d$, we say $S$ *is $\gamma$-thin at $x \in S$* if there is a sequence $r_i \downarrow 0$ such that

$$\Big(B(x, r_i) - \overline{B(x, r_i^\gamma)}\Big) \cap S = \phi \quad \text{for all } i. \tag{1.7}$$

$S_\gamma$ denotes the set of $\gamma$-thin points for $S$. A set is $\gamma$-thin if it is $\gamma$-thin at each of its points.

Note that $S_\gamma$ is indeed a $\gamma$-thin set and that $S_1 = S$. It is easy to use (1.5) to see that $S(X_t)_{\alpha/2} \subset C_\alpha(X_t)$ (see Corollary 4.4). The lower bound in (1.6) is then proved in Section 6 by constructing a random measure which is supported by the $\alpha/2$-thin points in $S(X_t)$, and an energy calculation which in fact will show $\dim\big(S(X_t)_{\alpha/2}\big) \geq \frac{8}{\alpha} - 2$ a.s. It is the existence of $\gamma$-thin subsets of positive Hausdorff dimension which invalidates the multifractal formalism of [Ha] and so we study the fractal properties of these sets in Section 2. Tribe (1991) showed a less stringent condition than (1.7) (with $S = S(X_t)$) holds for $X_t$ -a.a. $x$ (here $r_i^\gamma$ is replaced with $r_i/2$) to show that for $d \geq 3$, $X_t$ -a.a. $x$ are disconnected from the rest of $S(X_t)$.

Given the Hausdorff dimensions of $B_\alpha(X_t)$ and $C_\alpha(X_t)$, it is natural to ask about other fractal indices for these sets. These sets are a.s. dense in $S(X_t)$ for $2 \leq \alpha \leq 4$ and so box-counting gives

$$\Delta\big(B_\alpha(X_t)\big) = \Delta\big(C_\alpha(X_t)\big) = \Delta\big(S(X_t)\big) = 2 \quad \text{a.s.}, \quad 2 \leq \alpha \leq 4,$$

where

$$\Delta(E) = \lim_{k \to \infty} \log N(E, 2^{-k}) / \log(2^k)$$

and $N(E, 2^{-k})$ is the number of dyadic cubes of side $2^{-k}$ which intersect $E$. If Dim $(E)$ denotes the packing dimension of $E$ (the precise definition is recalled in Section 2), then the existence of an appropriate set of $\alpha/2$-thin points and a general result on fractal properties of these sets (Lemma 2.6) implies that

$$\mathrm{Dim}\big(B_\alpha(X_t)\big) \geq 4 - \alpha \quad \text{for } 2 \leq \alpha \leq 4. \tag{1.8}$$

Now $4 - \alpha > \frac{8}{\alpha} - 2$ for $2 < \alpha < 4$ and so $B_\alpha(X_t)$ and $C_\alpha(X_t)$ have distinct covering and packing indices in this range, which shows another way in which the multifractal formalism breaks down. This is discussed in Section 7 where we also calculate precise analogues of the mass exponents $b(q)$. (They will be piecewise linear functions.)

$c$, $c_n$, $c_{m,n}$ will denote constants whose value is unimportant and may change from line to line.

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## 2. Centered fractal measures with respect to $\mu$.

We will summarise definitions from Olsen [O] and refer the reader to that paper for proofs. Here $\mu$ is a fixed locally finite measure in $\mathbb{R}^d$ and $q$, $t$ are fixed real numbers.

$$\overline{\mathcal{H}}_\mu^{q,t}(E) = \liminf_{\delta \downarrow 0} \left\{ \sum_i \big(\mu(B(x_i, r_i))\big)^q (2r_i)^t \mid E \subset \bigcup_i B(x_i, r_i), x_i \in E, r_i < \delta \right\}. \tag{2.1}$$

This set function is not monotone in $E$, but

$$\mathcal{H}_\mu^{q,t}(E) = \sup_{F \subset E} \overline{\mathcal{H}}_\mu^{q,t}(F)$$

is a metric outer measure on subsets $E \subset \mathbb{R}^d$, known as the $(q,t)$ centered Hausdorff measure w.r.t. $\mu$ when $q = 0$, $\mathcal{H}_\mu^{0,t}$ differs from the usual $s^t$-Hausdorff measure by no more than a bounded factor. Since the measures are monotone in $t$ we can define

$$\begin{aligned} \dim_\mu^q(E) &= \inf\{t \in \mathbb{R} : \mathcal{H}_\mu^{q,t}(E) = 0\} \\ &= \sup\{t \in \mathbb{R} : \mathcal{H}_\mu^{q,t}(E) = +\infty\}. \end{aligned} \tag{2.2}$$

For any non-zero $\mu$ these 'multifractal dimensions' reduce to the standard Hausdorff or covering dimension when $q = 0$ (with the usual convention that $0^\circ = 1$). That is,

$$\dim(E) = \dim_\mu^0(E).$$

The corresponding packing functions are

$$\begin{aligned} \overline{\mathcal{P}}_\mu^{q,t}(E) &= \limsup_{\delta \downarrow 0} \left\{ \sum_i \big(\mu(B(x_i, r_i))\big)^q (2r_i)^t \mid B(x_i, r_i) \text{ disjoint, } x_i \in E, r_i < \delta \right\}; \\ \mathcal{P}_\mu^{q,t}(E) &= \inf_{E \subset \bigcup E_j} \sum_j \overline{\mathcal{P}}_\mu^{q,t}(E_j); \end{aligned} \tag{2.3}$$

$$\begin{aligned} \mathrm{Dim}_\mu^q(E) &= \inf\{t \in \mathbb{R} : \mathcal{P}_\mu^{q,t}(E) = 0\} \\ &= \sup\{t \in \mathbb{R} : \mathcal{P}_\mu^{q,t}(E) \equiv +\infty\}. \end{aligned} \tag{2.4}$$

As before we have

$$\mathrm{Dim}(E) = \mathrm{Dim}_\mu^0(E),$$

where Dim denotes the packing dimension as defined in Taylor, Tricot (1985) [TT]. For a summary of the properties of the standard Hausdorff and packing measures, see the book by Falconer (1990) [F].

There is one technical procedure which we need to extend from the standard $q = 0$ case first considered in [TT]. This allows us to replace packing with disjoint balls by packing with semi-dyadic cubes. For a positive integer $r$ we say that $D \in \mathcal{D}_r$ if, for some integers $k_i$

$$D = \prod_{i=1}^{d} \left[ \frac{1}{2} k_i 2^{-r}, \left( \frac{1}{2} k_i + 1 \right) 2^{-r} \right].$$

$\mathcal{D}_r^K$ is the set of these semi-dyadic cubes contained in $[-K, K]^d$. A cube $D \in \mathcal{D}_r$ is eligible to be used for packing $E \subset \mathbb{R}^d$ if $E$ intersects the unique $D^* \in \mathcal{D}_{r+1}$ such that $D^* \subset D$ and $D^*$ has the same center as $D$. If, in (2.3) we replace balls by such cubes $D_i \in \mathcal{D} = \bigcup_{r=1}^{\infty} \mathcal{D}_r$, and replace $2r_i$ by the diameter $d_i$ of $D_i$, we obtain a new set function $**\overline{\mathcal{P}}_\mu^{q,t}(E)$. For $q \leq 0$, this will be comparable to $\mathcal{P}_\mu^{q,t}$.

**Lemma 2.1**. For fixed $t \in \mathbb{R}$, $q \leq 0$ there are finite constants $c_{2.1}$, $c_{2.2}$ such that for every $\mu > 0$, $E \subset \operatorname{supp} \mu$

$$c_{2.1} \overline{\mathcal{P}}_\mu^{q,t}(E) \leq ** \overline{\mathcal{P}}_\mu^{q,t}(E) \leq c_{2.2} \overline{\mathcal{P}}_\mu^{q,t}(E). \tag{2.5}$$

**Proof**. (i) Suppose $\overline{\mathcal{P}}_\mu^{q,t}(E) = K > 0$. Then given $\delta > 0$, $\varepsilon > 0$ we can find a centered packing $B(x_i, r_i)$, $x_i \in E$, $r_i < \delta$ such that

$$\sum_i [\mu(B(x_i, r_i))]^q (2r_i)^t > K(1 - \varepsilon). \tag{2.6}$$

But every $B(x_i, r_i)$ contains an eligible semi-dyadic $D_i$ of diameter $d_i$ where

$$2r_i \geq d_i \geq c_1 r_i,$$

so that

$$d_i^t \geq c_{2.1} (2r_i)^t.$$

Since $D_i \subset B(x_i, r_i)$ and $q \leq 0$

$$\mu(D_i)^q \geq \left[\mu\big(B(x_i, r_i)\big)\right]^q.$$

Substituting in (2.6) gives

$$\sum_i [\mu(D_i)]^q (d_i)^t \geq c_{2.1} K(1 - \varepsilon),$$

so that

$$** \overline{\mathcal{P}}_\mu^{q,t}(E) \geq c_{2.1} K(1 - \varepsilon).$$

Since $\varepsilon$ is arbitrary the left inequality is established for $K$ finite. A similar argument is valid for $K = +\infty$, and there is nothing to prove if $K = 0$.

(ii) The right-hand inequality can be obtained by a similar argument, starting with a semi-dyadic packing of $E$ and replacing each $D_i$ by a centered ball $B(x_i, r_i) \subset D_i$ with

$$c_2 d_i \geq r_i \geq c_3 d_i.$$

$\square$

**Corollary 2.2**. If we define semi-dyadic packing measure by

$$**\mathcal{P}_\mu^{q,t}(E) = \inf_{E\subset\bigcup F_j} \sum_j **\overline{\mathcal{P}}_\mu^{q,t}(F_j),$$

then for $q \leq 0$,

$$c_{2.1}\mathcal{P}_\mu^{q,t}(E) \leq **\mathcal{P}_\mu^{q,t}(E) \leq c_{2.2}\mathcal{P}_\mu^{q,t}(E);$$

and the dimension indices for $**\mathcal{P}_\mu^{q,t}$ defined by (2.4) will be the same as those for $\mathcal{P}_\mu^{q,t}$.

**Remark 2.3**. We could define semi-dyadic covering measures $**\mathcal{H}_\mu^{q,t}$ and we would obtain analogous results to Lemma 2.1 and its Corollary whenever $q \leq 0$. The case $q = 0$ of all these results is well-known. We believe that the corresponding results for $q > 0$ (for either covering or packing) are false unless one imposes a strong regularity assumption on the measure $\mu$.

For a fixed $\mu$, Olsen [O] defines two functions (mass exponents)

$$\begin{aligned} b = b_\mu &: q \to \dim_\mu^q(\operatorname{supp}\mu), \\ B = B_\mu &: q \to \operatorname{Dim}_\mu^q(\operatorname{supp}\mu), \end{aligned} \tag{2.7}$$

defined for all $q \in \mathbb{R}$. He proves (Propositions 2.4, 2.10) that, for all $\mu > 0$, $q \in \mathbb{R}$

$$b(q) \leq B(q); \tag{2.8}$$

$$b(q) \text{ and } B(q) \text{ are decreasing ;} \tag{2.9}$$

$$B(q) \text{ is convex.} \tag{2.10}$$

For super-Brownian motion we will establish the values of $b(q)$ and $B(q)$ for all $q$ even though the results linking these functions to the multifractal spectrum of $\mu$ do not yield useful information because $\mu$ is not sufficiently regular at every point of its support.

We recall a method which is often used to obtain a lower bound for packing measure. Olsen [O], Theorem 2.15 gives a multifractal version of this, but we only need the standard version with $q = 0$ (which does not depend on $\mu$).

**Lemma 2.4**. There is a universal constant $c_{2.3}(d)$ such that if $\nu$ is a Borel measure with $\nu(E) > 0$, and $\liminf\limits_{r\downarrow 0} \frac{\nu(B(x,r))}{r^t} \leq 1$ for all $x \in E$, then

$$s^t - p(E) \equiv \mathcal{P}_\mu^{0,t}(E) \geq c_{2.3}\nu(E) > 0.$$

For a proof see, for example, [TT]. The next result is essentially due to Frostman (1935) [Fr]. A proof can be found in Falconer [F].

**Lemma 2.5**. Given a Borel set $E \subset \mathbb{R}^d$ such that

$$s^t - m(E) = \mathcal{H}_\mu^{0,t}(E) \geq 1,$$

there is a non-zero measure $\nu$ concentrated on $E$ such that

$$\nu\big(\overline{B(x,r)}\big) \leq c_{2.4}(2r)^t \text{ for all } x \in \mathbb{R}^d, 0 < r < 1.$$

Here $c_{2.4} = c_{2.4}(d)$ is a universal constant depending only on $d$.

Recall the definition of a $\gamma$-thin set from the Introduction. Subsets $A \subset \mathbb{R}^d$ which are $\gamma$-thin and have positive Hausdorff dimension must have a bigger packing dimension.

**Lemma 2.6**. For any $\gamma$-thin set $A \subset \mathbb{R}^d$

$$\text{Dim }(A) \geq \gamma \dim(A)$$

**Proof**. If $\dim A = 0$ there is nothing to prove. Suppose $\dim A = \beta > 0$. Then for $\eta > 0$, the Hausdorff measure,

$$s^{\beta-\eta} - m(A) = +\infty.$$

By Lemma 2.5 there is a measure $\nu$ concentrated on $A$ such that for all $x \in \mathbb{R}^d$, $0 < r < 1$,

$$\nu\big(B(x,r)\big) \leq r^{\beta-\eta}.$$

If $\{r_i\}$ is as in Definition 1.1, then

$$\nu\big(B(x,r_i) \cap A\big) = \nu\big(\overline{B(x,r_i^\gamma)} \cap A\big) \leq r_i^{\gamma(\beta-\eta)},$$

so that $\liminf\limits_{r \downarrow 0} \frac{\nu(B(x,r))}{r^{\gamma(\beta-\eta)}} \leq 1$ for all $x \in A$. By Lemma 2.4,

$$s^t - p(A) \geq c_{2.3}\nu(A) > 0$$

for $t = \gamma(\beta - \eta)$. Hence

$$\text{Dim}(A) \geq \gamma(\beta - \eta).$$

Since $\eta$ is arbitrary, $\text{Dim}(A) \geq \gamma\beta$. $\square$

**Corollary 2.7**. If $A \subset \mathbb{R}^d$ is $\gamma$-thin, then

$$\dim A \leq \frac{d}{\gamma}.$$

We will use $\gamma$-thin sets to relate the upper and lower indices, $\overline{d}(\mu, x)$ and $\underline{d}(\mu, x)$ for a Borel measure $\mu$ at $x$.

**Lemma 2.8**. For any Borel measure $\mu$, suppose $A$ is a $\gamma$-thin subset of $S(\mu)$, and

$$\underline{d}(\mu, x) \geq \alpha \quad \text{for all } x \in A.$$

Then

$$\overline{d}(\mu, x) \geq \gamma\alpha \quad \text{for all } x \in A.$$

**Proof**. For each $\eta > 0$, $x \in A$, $\underline{d}(\mu, x) \geq \alpha$ implies that

$$\lim_{r \downarrow 0} \frac{\mu\big(B(x,r)\big)}{r^{\alpha-\eta}} = 0,$$

and so

$$\mu\big(B(x,r)\big) \leq r^{\alpha-\eta} \quad \text{for } 0 < r \leq r_0 = r_0(x).$$

For $\{r_i\}$ as in Definition 1.1, and $i$ large enough

$$\mu\big(B(x,r_i)\big) = \mu\big(\overline{B(x,r_i^{\gamma})}\big) \leq r_i^{\gamma(\alpha-\eta)},$$

so that

$$\liminf_{r \downarrow 0} \frac{\mu\big(B(x,r)\big)}{r^{\gamma(\alpha-\eta)}} \leq 1,$$

which implies

$$\overline{d}(\mu, x) = \limsup_{r \downarrow 0} \frac{\log \mu(B(x,r))}{\log r} \geq \gamma(\alpha - \eta).$$

Since $\eta$ is arbitrary, for all $x \in A$,

$$\overline{d}(\mu, x) \geq \gamma\alpha.$$

□

**3. Preliminaries on Super-Brownian Motion**.

Let $\mathbb{P}_{m_0}^{\gamma_0}$ be the law of $d$-dimensional ($d \geq 3$) super-Brownian motion on the canonical space $(\Omega_X, \mathcal{F}_X)$ of continuous $M_F(\mathbb{R}^d)$-valued paths on $\mathbb{R}_+$ with its Borel $\sigma$-field. Here $\gamma_0 > 0$ is the branching rate and $m_0 \in M_F(\mathbb{R}^d)$ is the initial state. Let $X_t(\omega) = \omega(t)$ denote the coordinate mappings on $\Omega_X$. If $\phi : \mathbb{R}^d \to \mathbb{R}_+$ is bounded and measurable, $U_t^{\gamma_0}\phi(x)$ denotes the unique solution (of the weak form) of

$$\text{(3.1)} \qquad \frac{\partial U_t}{\partial t}(x) = \frac{\Delta}{2} U_t(x) - \frac{\gamma_0}{2} U_t(x)^2; \quad U_0(x) = \phi(x).$$

Then $X$ is the unique $M_F(\mathbb{R}^d)$-valued diffusion such that (write $\mu(f)$ for $\int f d\mu$) for $\phi$ as above

$$\text{(3.2)} \qquad \mathbb{P}_{m_0}^{\gamma_0}\Big(\exp\big(-X_t(\phi)\big)\Big) = \exp\big(-m_0(U_t^{\gamma_0}\phi)\big).$$

The scaling properties of (3.1) show that

$$\text{(3.3)} \qquad \mathbb{P}_{m_0}^{\gamma_0}(X \in \cdot) = \mathbb{P}_{m_0/\gamma_0}^{1}(\gamma_0 X \in \cdot)$$

and so all the results in this paper will follow for general $\gamma_0$ once they are established for $\gamma_0 = 4$, a value which arises naturally from Le Gall's representation of $X$ in terms of a path-valued process. Henceforth $\gamma_0 = 4$ and dependence on $\gamma_0$ is suppressed in our notation. One easily checks that for $\phi \equiv \lambda$ $\big(\lambda \in [0,\infty)\big)$,

$$U_t\lambda(x) = \lambda(1+2\lambda t)^{-1} \tag{3.4}$$

and so $X_t(1)$ is Feller's continuous state branching process scaled so that

$$X_t(1) = m_0(1) + \int_0^t 2\sqrt{X_s(1)}dB_s \tag{3.5}$$

where $B$ is a one-dimensional Brownian motion (see Knight (1981), p. 100). More information on superprocesses may be found in Dawson (1992, 1993).

The proof in Section 4 will use Le Gall's path-valued process and so we now review some results from Le Gall (1993, 1994a) [LG1, LG2]. The path-valued process takes values in

$$\mathcal{W} = \big\{(w,\zeta) \in C(\mathbb{R}_+,\mathbb{R}^d) \times \mathbb{R}_+ : w(s) = w(\zeta) \text{ for } s \geq \zeta\big\}$$

which is metrized by $p\big((w,\zeta),(w',\zeta')\big) = \|w - w'\|_\infty + |\zeta - \zeta'|$. We will usually write $w$ for $(w,\zeta)$, as the lifetime $\zeta$ of $w$ will usually be clear from $w$ and the context. $(\Omega_W, \mathcal{F}_W)$ denotes the space of continuous $\mathcal{W}$-valued paths on $\mathbb{R}_+$ with its Borel $\sigma$-field. $(W_s, s \geq 0)$ are coordinate variables on $\Omega_W$, $\zeta_s$ is the lifetime of $W_s$ and $\hat{W}_s = W_s(\zeta_s)$ is the endpoint of $W_s$. For $w$ in $\mathcal{W}$, we abuse notation slightly and write $\mathbb{P}_w$ for the law on $(\Omega_W, \mathcal{F}_W)$ of the path-valued process, starting at $w$, and associated with $d$-dimensional Brownian motion. Under $P_w$, $W.$ is a $\mathcal{W}$-valued diffusion and $\zeta.$ is a one-dimensional reflecting Brownian motion. Let $\{L_s^x : s \geq 0\}$ denote the local time of $\zeta$, normalized so that it is an occupation density. The constant path $\underline{x}(s) = x$ with zero lifetime is regular for $W.$ (since 0 is regular for $\zeta.$) and so we may define the Itô excursion measure, $\mathbb{N}_x$, of excursions of $W$ from $\underline{x}$. $\mathbb{N}_x$ is a $\sigma$-finite measure on $(\Omega_W, \mathcal{F}_W)$ normalized so that it is the intensity of the Poisson measure of excursions, $\Pi_x$, of $W$ from $\underline{x}$ completed up to time $\tau = \inf\{t : L_t^0 = 1\}$. Then $\mathbb{N}_x(\zeta. \in \cdot)$ is the excursion measure for the reflecting Brownian motion $\zeta.$ normalised so that

$$\mathbb{N}_x(\sup\nolimits_s \zeta_s > h) = (2h)^{-1}. \tag{3.6}$$

Define continuous $M_F(\mathbb{R}^d)$-valued processes by

$$X_t(\phi) = \int_0^\tau \phi(\hat{W}_s)d_sL_s^t \text{ and } \tilde{X}_t(\phi) = \int_0^\sigma \phi(\hat{W}_s)d_sL_s^t$$

where $\sigma(W) = \{s > 0 : \zeta_s = 0\}$ and $\tau$ is as above. Then (Theorem 2.1 of [LG1]), as the notation suggests,

$$\mathbb{P}_{\underline{x}}(X \in \cdot) = \mathbb{P}_{\delta_x}(\cdot).$$

Decompose $L_\cdot^t$ according to the excursions of $\zeta$ away from zero to see that for $t > 0$

$$X_t = \int\limits_{\Omega_W} \tilde{X}_t(W) d\Pi_x(W) \quad \mathbb{P}_{\underline{x}}\text{-a.s.} \tag{3.7}$$

and so

$$\mathbb{P}_{\delta_x}\Big(\exp\big(-X_t(\phi)\big)\Big) = \exp\Big(-\int 1 - \exp\big(-\tilde{X}_t(\phi)\big)\mathbb{N}_x(dW)\Big) = \exp\{-U_t\phi(x)\}. \tag{3.8}$$

The integral in (3.7) is a finite sum and implies (recall (3.6) as well)

$$\begin{aligned} &\mathbb{P}_{\delta_x}(X_t \in \cdot) = \mathbb{P}\left(\sum_{i=1}^N X_t^{(i)} \in \cdot\right) \quad \text{where } \{X_\cdot^{(i)} : i \in \mathbb{N}\} \quad \text{are i.i.d.} \\ &\text{and distributed as } \mathbb{N}_x(X_t \in \cdot \mid X_t \neq 0), \text{ and } N \text{ is} \\ &\text{an independent Poisson random variable with mean } (2t)^{-1}. \end{aligned} \tag{3.9}$$

Finally we let $\hat{\mathbb{P}}_w(\cdot) = \mathbb{P}_w(W_{\cdot\wedge\sigma} \in \cdot)$ be the law of the stopped path-valued process.

In Sections 5 and 6 we will use historical Brownian motion, $H_t$, which is the $M_F\big(C(\mathbb{R}_+, \mathbb{R}^d)\big)$-valued diffusion defined by (under $\mathbb{P}_{\underline{x}}$)

$$H_t(\phi) = \int\limits_0^\tau \phi(W_s) d_s L_s^t. \tag{3.10}$$

More precisely this defines historical Brownian motion starting at $\delta_{\underline{x}}$ at time 0 (see Le Gall (1994b) [LG3]). In general, historical Brownian motion is a time-inhomogeneous diffusion whose existence and abstract properties follow from the general theory of superprocesses by taking the spatial motion process to be $t \to B^t \equiv B(\cdot\wedge t)$ where $B$ is a Brownian motion. [DP] gives an introduction to historical processes. Let $\mathbb{Q}_{s,m_0}$ denote the law of historical Brownian motion starting at time $s \geq 0$ in state $m_0$. Here $m_0$ is a finite measure on $C(\mathbb{R}_+, \mathbb{R}^d)$ such that $y = y^s$ $m_0$-a.e. $\mathbb{Q}_{s,m_0}$ is a probability on $\big(\Omega_H, \mathcal{F}_H[s,\infty)\big)$, the space of continuous $M_F\big(C(\mathbb{R}_+, \mathbb{R}^d)\big)$-valued paths on $\mathbb{R}_+$ with the Borel $\sigma$-field generated by $\{H_u : u \geq s\}$. As (3.10) suggests, in general under $\mathbb{Q}_{0,m_0} \equiv \mathbb{Q}_{m_0}$, $X_t(\cdot) = H_t(\{y : y_t \in \cdot\})$ is a super-Brownian motion starting at $m_0$. Here we have identified $m_0$ with the obvious measure on $\mathbb{R}^d$. We frequently will use the fact that if $c > 2$, there is a random $\delta > 0$ $\mathbb{Q}_{m_0}$-a.s. such that for all $y$ in $\overline{\bigcup_{t\geq 0} S(H_t)}$ if $|s-t| \leq \delta$, then $|y_s - y_t| \leq c\sqrt{|t-s|\log^+ 1/|t-s|}$ (see Theorem 8.7 of [DP]). This uniform Lévy modulus for $H$ gives us uniform control over the rate of propagation of $H$.

In addition we often use the equivalence of the laws $\mathbb{P}_{m_0}(X_s \in \cdot)$ and $\mathbb{P}_{m_1}(X_t \in \cdot)$ for $m_0$, $m_1 \in M_F(\mathbb{R}^d) - \{0\}$, $s, t > 0$ (see Evans-Perkins (1990) [EP]) to reduce to the case $t = 1$ and $m_0 = \delta_0$.

## 4. Uniform Upper Density Results for Super-Brownian Motion.

In this section we prove (1.5). More precisely we show (recall $d \geq 3$)

**Theorem 4.1.** If $\{X_t : t \geq 0\}$ is super-Brownian motion starting at $m_0$ under $\mathbb{P}_{m_0}$, then

$$\liminf_{r \downarrow 0} \frac{\log\Big(X_t\big(B(x,r)\big)\Big)}{\log r} = 2 \quad \forall x \in S(X_t) \quad \mathbb{P}_{m_0}\text{-a.s.} \quad \forall t > 0.$$

This is an immediate consequence of the following two results.

**Proposition 4.2.** There is a constant $c_{4.1} = c_{4.1}(d)$ such that for $m_0 \in M_F(\mathbb{R}^d)$

$$\limsup_{r \downarrow 0} \sup_x \frac{X_t\big(B(x,r)\big)}{r^2 \log 1/r} \leq c_{4.1} \quad \mathbb{P}_{m_0}\text{-a.s.} \quad \forall t > 0.$$

This is immediate from Theorem 4.7 of [BEP].

**Notation.** $h(t) = 5(t \log^+ 1/t)^{1/2}$, where $\log^+ x = |\log x| \vee 1$.

**Proposition 4.3.** For any $m_0$ in $M_F(\mathbb{R}^d)$ and any $t > 0$

$$\overline{\lim_{r \downarrow 0}} \frac{X_t\big(B(x,r)\big)}{r^2} (\log^+ 1/r)^{3+\eta} = \infty \quad \forall x \in S(X_t) \quad \forall \eta > 0 \quad \mathbb{P}_{m_0}\text{-a.s.}$$

**Proof.** By [EP] we may assume $m_0 = \delta_0$ and $t = 1$. Let $\eta > 0$ and define $g(r) = r^2(\log^+ 1/r)^{-3-\eta}$. If $\delta \in [0, \infty]$, let

$$K(\delta) = \{y \in C(\mathbb{R}_+, \mathbb{R}^d) : |y(t) - y(s)| \leq \frac{1}{2} h(|t-s|) \quad \forall |t-s| < \delta, \quad t, s \geq 0\}$$

and

$$\Omega(\delta) = \{W \in \Omega_W : W_t \in K(\delta) \quad \forall t \geq 0\}.$$

For $y \in C(\mathbb{R}_+, \mathbb{R}^d)$ (respectively, $W \in \Omega_W$) let $\delta(y)$ (respectively, $\delta(W)$) be the largest $\delta$ such that $y \in K(\delta)$ (respectively, $W \in \Omega(\delta)$). Note that such maximal $\delta$'s do exist (but may be 0!). The uniform Lévy modulus for the historical process described in the last section (Theorem 8.7 of [DP]) and the fact that $\{W_s : 0 \leq s \leq \tau\}$ is the closure of $\bigcup_{t \geq 0} S(H_t)$ ([LG3]) imply that $p(\delta) \equiv \mathbb{P}_0\big(\Omega(\delta)^c\big) \downarrow 0$ as $\delta \downarrow 0$. Since $p(\delta) = 1 - \exp\{-\mathbb{N}_0\big(\Omega(\delta)^c\big)\}$, we see that

$$\mathbb{N}_0\big(\Omega(\delta)^c\big) \downarrow 0 \text{ as } \delta \downarrow 0. \tag{4.1}$$

We first consider $\tilde{X}$, under $\mathbb{N}_0$ and $\hat{\mathbb{P}}_y$. Fix $N \in \mathbb{N}$ and $y \in K(2^{-N})$. By Proposition 2.5 of [LG2] there is a Poisson point process $\mathcal{M}_1$ on $[0,1] \times \Omega_W$ with intensity $2dt\mathbb{N}_{y(t)}(dW)$ such that

$$\tilde{X}_1 = \int_0^1 \int_\Omega \tilde{X}_{1-t}(W') \mathcal{M}_1(dt, dW') \quad \hat{\mathbb{P}}_y\text{-a.s.}$$

This decomposes $\tilde{X}_1$ into clusters which branch off $y$ at time $t$ as $W$ "backs down" $y$. Assume $W \in \Omega(2^{-N})$ and $k \geq N (k \in \mathbb{N})$. Then $\mathcal{M}_1(W)$ is supported on $[0,1] \times \Omega(2^{-N})$ (see the explicit description of these points as excursions of the lifetime process above its current minimum in Section 2 of [LG2]). Let $(t, W') \in S(\mathcal{M}_1)$ satisfy $1 - h^{-1}(2^{-k}) < t$. Then

$$
\begin{aligned}
\tilde{X}_{1-t}(W')\Big(B\big(y(1), 2^{-k}\big)\Big) &= \int_0^\sigma 1(|W'_s(1-t) - y(1)| < 2^{-k}) d_s L_s^{1-t} \\
&\geq \int_0^\sigma 1(|W'_s(1-t) - W'_s(0)| + |y(t) - y(1)| < 2^{-k}) d_s L_s^{1-t} \\
&\geq \int_0^\sigma 1\big(h(1-t) < 2^{-k}\big) d_s L_s^{1-t} \quad (\text{because } y \in K(2^{-N}) \text{ and } W' \in \Omega(2^{-N})) \\
&= \tilde{X}_{1-t}(W')(1).
\end{aligned}
$$

Therefore we have

(4.2)

$$
\int_{1-h^{-1}(2^{-k})}^{1} \int \tilde{X}_{1-t}(W')(1) \mathcal{M}_1(dt, dW') \leq \tilde{X}_1\Big(B\big(y(1), 2^{-k}\big)\Big) \text{ for } k \geq N \text{ and } W \in \Omega(2^{-N}).
$$

Let $n > N$ and $\lambda_k = \varepsilon_k g(2^{-k})^{-1}$ where $\{\varepsilon_k\}$ is non-negative and summable. Then

$$
\begin{aligned}
&\hat{\mathbb{P}}_y \left(\delta(W) \geq 2^{-N}, \tilde{X}_1\Big(B\big(y(1), 2^{-k}\big)\Big) \leq g(2^{-k}) \text{ for all } N \leq k \leq n\right) \\
&\leq \hat{\mathbb{P}}_y \left( \int_{1-h^{-1}(2^{-k})}^{1-h^{-1}(2^{-k-1})} \int \tilde{X}_{1-t}(1) \mathcal{M}_1(dt, dW) \leq g(2^{-k}) \text{ for all } N \leq k < n \right) \\
&\qquad \hat{\mathbb{P}}_y \left( \int_{1-h^{-1}(2^{-n})}^{1} \int \tilde{X}_{1-t}(1) \mathcal{M}_1(dt, dW) \leq g(2^{-n}) \right) \\
&\leq \prod_{k=N}^{n-1} e^{\varepsilon_k} \exp\left\{ - \int_{1-h^{-1}(2^{-k})}^{1-h^{-1}(2^{-k-1})} \int 1 - e^{-\tilde{X}_{1-t}(\lambda_k)} \mathbb{N}_{y(t)}(dW) 2dt \right\} \\
&\qquad \times e^{\varepsilon_n} \exp\left\{ - \int_{1-h^{-1}(2^{-n})}^{1} \int 1 - e^{-\tilde{X}_{1-t}(\lambda_n)} \mathbb{N}_{y(t)}(dW) 2dt \right\}
\end{aligned}
$$

$$\leq c\exp\left\{-\sum_{k=N}^{n-1}\int\limits_{h^{-1}(2^{-k-1})}^{h^{-1}(2^{-k})}\frac{2\lambda_k}{1+2\lambda_k t}\,dt-\int\limits_0^{h^{-1}(2^{-n})}\frac{2\lambda_n}{1+2\lambda_n t}\,dt\right\}\ \text{(by (3.8) and (3.4))}$$

$$\leq c\left[\prod_{k=N}^{n-1}\frac{1+2\varepsilon_k g(2^{-k})^{-1}h^{-1}(2^{-k-1})}{1+2\varepsilon_k g(2^{-k})^{-1}h^{-1}(2^{-k})}\right]\Big(1+2\varepsilon_n g(2^{-n})^{-1}h^{-1}(2^{-n})\Big)^{-1}.$$

An elementary argument shows that the product in square brackets is bounded by

$$c\prod_{k=N}^{n-1}h^{-1}(2^{-k-1})/h^{-1}(2^{-k})=ch^{-1}(2^{-n})\big(h^{-1}(2^{-N})\big)^{-1}$$

provided that

$$\sum_N^\infty g(2^{-k})2^{2k}k\varepsilon_k^{-1}<\infty. \tag{4.3}$$

Assuming (4.3) we see that for $y\in K(2^{-N})$,

$$\begin{aligned}&\hat{\mathbb{P}}_y\left(\delta(W)\geq 2^{-N},\tilde{X}_1\Big(B\big(y(1),2^{-k}\big)\Big)\leq g(2^{-k})\text{ for all }N\leq k\leq n\right)\\&\quad\leq c_N h^{-1}(2^{-n})\left(2\varepsilon_n g(2^{-n})^{-1}h^{-1}(2^{-n})\right)^{-1}\\&\quad\leq c_N g(2^{-n})\varepsilon_n^{-1}.\end{aligned}\tag{4.4}$$

Let

$$\Lambda_N=\{x\in S(\tilde{X}_1):\tilde{X}_1\big(B(x,(\sqrt{d}+1)2^{-k})\big)\leq g(2^{-k})\quad\forall k\geq N\},$$

and for $n>N$ and $C\in\mathcal{D}_n$, set

$$T_C=\inf\{t:\zeta_t=1,\hat{W}_t\in C\}.$$

If $C\in\mathcal{D}_n$ and $x\in C\cap\Lambda_N$ then $T_C<\infty$ and $B(\hat{W}_{T_C},2^{-k})\subset B\big(x,(\sqrt{d}+1)2^{-k}\big)$ for $k\leq n$ and so $\tilde{X}_1\big(B(\hat{W}_{T_C},2^{-k})\big)\leq g(2^{-k})$ for $N\leq k\leq n$. Let $\{\theta_t\}$ be the canonical shift operators on $\Omega_W$. Then for $C\in\mathcal{D}_n$, the strong Markov property under $\mathbb{N}_0$ (see Section 2 of [LG2]) shows that

$$\begin{aligned}&\mathbb{N}_0\left(\delta(W)\geq 2^{-N},C\cap\Lambda_N\neq\phi\right)\\&\leq\mathbb{N}_0\Big(\delta(W_{\cdot\wedge T_C})\geq 2^{-N},T_C<\infty,\delta\circ\theta_{T_C}\geq 2^{-N},\tilde{X}_1\circ\theta_{T_C}\big(B(\hat{W}_{T_C},2^{-k})\big)\leq g(2^{-k})\text{ for }N\leq k\leq n\Big)\\&\leq\mathbb{N}_0\Big(1\left(\delta(W_{\cdot\wedge T_C})\geq 2^{-N},T_C<\infty\right)\hat{\mathbb{P}}_{W_{T_C}}\left(\delta\geq 2^{-N},\tilde{X}_1\left(B\left(W_{T_C}(1),2^{-k}\right)\right)\leq g(2^{-k})\text{ for }N\leq k\leq n\right)\Big)\\&\leq\mathbb{N}_0(T_C<\infty)c_N g(2^{-n})\varepsilon_n^{-1}\quad\text{(by (4.4))}\\&\leq c_N 2^{-n(d-2)}g(2^{-n})\varepsilon_n^{-1}.\end{aligned}$$

In the last line we used Theorem 3.1 of Dawson et al (1989) [DIP] to bound $\mathbb{P}_{\delta_0}\big(X_1(C)>0\big)=1-\exp\{-\mathbb{N}_0(T_C<\infty)\}$ by $c2^{-n(d-2)}$. Sum the above inequality over $C$ in $\mathcal{D}_n^K$ to conclude

$$\mathbb{N}_0\big(\delta(W)\geq 2^{-N},\Lambda_N\cap[-K,K]^d\neq\phi\big)\leq c_{N,K}2^{2n}g(2^{-n})\varepsilon_n^{-1}\to 0 \text{ as } n\to\infty \text{ (by 4.3).}$$

In view of (4.1) and the fact that $\Lambda_N$ increases with $N$, this implies $\Lambda_N=\phi$ $\mathbb{N}_0$-a.e. and therefore

$$\overline{\lim_{r\downarrow 0}}\,\frac{\tilde{X}_1\big(B(x,r)\big)}{g(r)}\geq c>0\quad \forall x\in S(\tilde{X}_1)\quad \mathbb{N}_0\text{-a.e.}$$

As this is valid for any $\eta>0$, one sees easily that $c$ may be taken to be $\infty$ in the above. Finally note that if $\varepsilon_k=k^{-1-\eta/2}$, then (4.3) holds. This proves the result with $\mathbb{N}_0$ in place of $\mathbb{P}_{\delta_0}$. The "cluster decomposition" (3.9) now gives the required result under $\mathbb{P}_{\delta_0}$. □

**Corollary 4.4**. For each $\gamma>1$ $S(X_t)_\gamma\subset C_{2\gamma}(X_t)$ $\mathbb{P}_{m_0}$-a.s. $\forall t>0$.

**Proof**. Theorem 4.1 shows that the hypotheses of Lemma 2.8 hold with $A=S(X_t)_\gamma$ and $\alpha=2$. Lemma 2.8 gives the above inclusion. □

## 5. Upper Bound for the Hausdorff Dimension of Points of Small Local Density.

In this section we prove:

**Proposition 5.1**. For $m_0\in M_F(\mathbb{R}^d)$ and $t>0$ $\mathbb{P}_{m_0}$-a.s. we have

$$\dim\big(B_\alpha(X_t)\big)\leq\dim\big(C_\alpha(X_t)\big)\leq\frac{8}{\alpha}-2\quad\text{for } 2\leq\alpha\leq 4,$$

and $B_\alpha(X_t)=C_\alpha(X_t)=\phi$ for $\alpha>4$.

**Remark 5.2**. Theorem 4.1 implies $B_\alpha(X_t)=\phi$ and $C_\alpha(X_t)=S(X_t)$ for $\alpha<2$.

**Proof**. We will work with the historical process $H_t$ on its canonical space with law $\mathbb{Q}_{m_0}$. By [EP] we may assume $t=1$ and $m_0=\delta_0$. It clearly suffices to prove the result for a fixed $\alpha>2$ (the result is trivial if $\alpha=2$). Choose $2<\beta<\alpha$. Call a semi-dyadic cube $D\in\mathcal{D}_n$ $\beta$-light if $X_1(D^*)>0$ (recall $D^*$ is the unique set in $\mathcal{D}_{n+1}$, with the same center as $D$) and $X_1(D)\leq 2^{-n\beta}$. Let $L_n^K(\beta)$ be the number of $\beta$-light semi-dyadic cubes in $\mathcal{D}_n^K$ and $\mathcal{L}_n^K(\beta)$ be the set of such cubes. Lemma 3.4 of [LPT] implies that

$$\mathbb{Q}_{\delta_0}\big(L_n^K(\beta)\big)\leq c_{5.1}(K)2^{-n(\beta-4)}.\tag{5.1}$$

Lemma 3.4 of [LPT] uses $\mathbb{N}_0$ instead of $\mathbb{Q}_{\delta_0}$, but since $X_1$ is a Poisson superposition of i.i.d. clusters with law $\mathbb{N}_0(X_1\in\cdot\mid X_1\neq 0)$ (see (3.9)), the result follows under $\mathbb{P}_{\delta_0}$ and hence $\mathbb{Q}_{\delta_0}$. If $x\in C_\alpha(X_1)\cap[-K,K]^d$ we may choose $1\geq r_i\downarrow 0$ such that $\lim_{i\to\infty}X_1\big(B(x,r_i)\big)r_i^{-\beta}=0$. For each $B(x,r_i)$, pick the largest semi-dyadic cube $D_i\subset B(x,r_i)\subset[-K-1,K+1]^d$ such that $x\in D_i^*$. If $D_i\in\mathcal{D}_{n_i}^{K+1}$ then for $i\geq i_0$, $D_i$

will be $\beta$-light because $X_1(D_i^*) > 0$ and $X_1(D_i) \le X_1\big(B(x,r_i)\big) < 2^{-n_i\beta}$ for $i$ large. A Borel-Cantelli argument and (5.1) show that

$$L_n^{K+1}(\beta) \le n^2 2^{n(4-\beta)} \quad \text{for } n \ge n_0(\omega) \quad \text{a.s.} \tag{5.2}$$

It follows that w.p. 1 for $K \in \mathbb{N}$ and $n \ge n_0(\omega)$

$$\begin{aligned} &C_\alpha(X_1) \cap [-K,K]^d \text{ can be covered, for any } N \text{ in } \mathbb{N}, \text{ by cubes } D^* \text{ where} \\ &D \text{ is in } \bigcup_{n=N}^{\infty} \mathcal{L}_n^{K+1}(\beta), \text{ and } L_n^{K+1}(\beta) \le n^2 2^{n(4-\beta)}. \end{aligned} \tag{5.3}$$

If $\alpha > 4$ we may choose $\beta > 4$, and (5.3) then implies $C_\alpha(X_1) = \phi$ a.s. Assume now $2 < \alpha < 4$, as the result for $\alpha = 4$ then follows. An easy calculation shows that (5.3) only implies $\dim\big(C_\alpha(X_1)\big) \le 4 - \alpha$ a.s. and so we must find a better cover of $C_\alpha(X_1)$. If $D^*$ is as in (5.3), we will cover $D^* \cap S(X_1)$ more economically by a union of much smaller balls.

**Lemma 5.3**. If $\rho > \beta$ then $\mathbb{Q}_{\delta_0}$-a.s. for sufficiently large $n$, for each $\beta$-light $D$ in $\mathcal{D}_n^n$, $S(X_1) \cap D^*$ may be covered by a union of $[2^{n(\rho-\beta)+3}]$ balls of radius $h(2^{-n\rho})$.

**Proof**. If $D \in \mathcal{D}_n$, let $\tilde{D}$ be the closed cube with sides parallel to those of $D$ such that $\tilde{D}$ has the same center as $D^*$ (and $D$), $D^* \subset \tilde{D} \subset D$, and the distance from $D^*$ to $\tilde{D}^c$, and from $\tilde{D}$ to $D^c$ is $2^{-n-3}$. Let $\varepsilon_n = 2^{-\rho n}$ and let $N_n(D^*)$ denote the number of points in the support of $H_1(\{y : y_{1-\varepsilon_n} \in \cdot\})$ which lie in $\tilde{D}$. The Lévy modulus for $\bigcup_{t\ge 0} S(H_t)$ (Theorem 8.7 of [DP]) shows that for $n \ge n_0(\omega)$, $S(X_1) \cap D^*$ is contained in the union of $N_n(D^*)$ open balls of radius $h(\varepsilon_n)$ centered at those points in the support of $H_1(\{y : y_{1-\varepsilon_n} \in \cdot\})$ which lie in $\tilde{D}$ (we are also using Proposition 8.11 of [DP]). It therefore remains to show

$$N_n(D^*) \le 2^{n(\rho-\beta)+3} \text{ for each } \beta\text{-light } D \text{ in } \mathcal{D}_n^n \text{ for } n \text{ large a.s.} \tag{5.4}$$

By Proposition 3.5 of [DP], conditional on $\mathcal{F}_{1-\varepsilon_n}^H = \sigma(H_s : s \le 1-\varepsilon_n)$, $N_n(D^*)$ has a Poisson law with mean $\lambda_n = X_{1-\varepsilon_n}(\tilde{D})(2\varepsilon_n)^{-1}$. We need a standard estimate on the tail of a Poisson law.

**Lemma 5.4**. If $N$ is a Poisson random variable with mean $\lambda$, then for $a > e$ and $M > 0$,

$$P\big(N \ge a(\lambda \vee M)\big) \le \exp\{-(\ln a - 1)a(\lambda \vee M)\}.$$

**Proof**. If $t = \ln\big(a(\lambda \vee M)/\lambda\big)$, then

$$\begin{aligned} P\big(N \ge a(\lambda \vee M)\big) &\le P(e^{tN}) \exp\{-ta(\lambda \vee M)\} \\ &= \exp\{\lambda\big(a(\lambda \vee M)\lambda^{-1} - 1\big) - \ln\big(a(\lambda \vee M)\lambda^{-1}\big)a(\lambda \vee M)\} \\ &\le \exp\big(a(\lambda \vee M) - (\ln a)a\lambda \vee M\big). \square \end{aligned}$$

If $\eta = \rho - \beta$ and we set $a = 4$ and $M = 2^{\eta n}$ in the above we obtain

$$\begin{aligned} &\mathbb{Q}_{\delta_0}\left(N_n(D^*) \ge 4(X_{1-\varepsilon_n}(\tilde{D})(2\varepsilon_n)^{-1} \vee 2^{\eta n}) \text{ for some } D \text{ in } \mathcal{D}_n^n \,\middle|\, \mathcal{F}_{1-\varepsilon_n}^H\right) \\ &\quad \le cn^d 2^{nd} \exp\{-4(\ln 4 - 1)2^{\eta n}\}. \end{aligned}$$

This is summable in $n$ and so by the Borel-Cantelli Lemma we see that

$$N_n(D^*) \le \big(2X_{1-\varepsilon_n}(\tilde{D})2^{n\rho}\big) \vee 2^{\eta n+2} \text{ for all } D \text{ in } \mathcal{D}_n^n \text{ for } n \text{ large } \mathbb{Q}_{\delta_0}\text{-a.s.} \tag{5.5}$$

If $P_x$ is the law of the total mass process $M_t = X_t(1)$ starting with $M_0 = x$, then for $D$ in $\mathcal{D}_n^n$

$$\begin{aligned}&\mathbb{Q}_{\delta_0}\left(X_{1-\varepsilon_n}(\tilde{D}) > 2^{2-n\beta}, H_1(\{y : y_{1-\varepsilon_n} \in \tilde{\mathcal{D}}\}) \le 2^{-n\beta} \mid \mathcal{F}^H_{1-\varepsilon_n}\right)\\ &\quad \le P_{2^{2-n\beta}}(M_{\varepsilon_n} \le 2^{-n\beta}) \quad \text{(e.g. by Proposition 3.2 of Perkins (1992)).}\end{aligned}$$

As $M_t$ is Feller's continuous state branching process (recall (3.5)), it is easy to bound the above probability by $\exp\{-2^{n\eta}\}$ using the explicit formulae for its Laplace transform, $P_x(e^{-\lambda M_t}) = \exp\{-\lambda x(1+2\lambda t)^{-1}\}$ (e.g. see Knight (1981, p. 100) and recall our branching rate is $\gamma_0 = 4$). To see this, use $P_x(M_{\varepsilon_n} \le b) \le P_x(e^{-\lambda M_{\varepsilon_n}})e^{\lambda b}$ and take $\lambda = 2^{n\rho}$, $x = 2^{2-n\beta}$ and $b = 2^{-n\beta}$. A Borel-Cantelli argument implies

$$\begin{aligned}&H_1\left(\left\{y : y_{1-\varepsilon_n} \in \tilde{D}\right\}\right) \le 2^{-n\beta} \text{ implies } X_{1-\varepsilon_n}(\tilde{D}) \le 2^{2-n\beta}\\ &\text{for all } D \text{ in } \mathcal{D}_n^n \text{ and } n \text{ large } \mathbb{Q}_{\delta_0}\text{-a.s.}\end{aligned} \tag{5.6}$$

Another application of the uniform Lévy modulus for paths in $\bigcup_{t\ge 0} S(H_t)$ implies (note that $h(\varepsilon_n) < 2^{-n-3}$ for $n$ large)

$$H_1\left(\left\{y : y_{1-\varepsilon_n} \in \tilde{D}\right\}\right) \le X_1(D) \text{ for all } D \text{ in } \mathcal{D}_n^n \text{ and } n \text{ sufficiently large.}$$

Hence (5.6) implies

$$X_1(D) \le 2^{-n\beta} \text{ implies } X_{1-\varepsilon_n}(\tilde{D}) \le 2^{2-n\beta} \quad \forall D \in \mathcal{D}_n^n \text{ and } n \text{ large } \mathbb{Q}_{\delta_0}\text{-a.s.}$$

Use this in (5.5) to see that (5.4) holds and this completes the proof of Lemma 5.3. □

To complete the proof of Proposition 5.1, take $\rho > \beta$ and $n_0 \ge K$ sufficiently large such that (5.3) and the conclusion of Lemma 5.3 hold for $n \ge n_0$. For each of the $\beta$-light cubes $D$ as in (5.3) ($D \in \mathcal{L}_n^{K+1}(\beta)$ with $n \ge N > n_0$) cover $D^* \cap S(X_1)$ as in Lemma 5.3. Then

$$\begin{aligned} s^t - m\left(C_\alpha(X_1) \cap [-K,K]^d\right) &\le \lim_{N\to\infty} \sum_{n\ge N} h(2^{-n\rho})^t n^2 2^{n(4-\beta)} 2^{3+n(\rho-\beta)}\\ &\le \lim_{N\to\infty} c \sum_{n\ge N} 2^{-n(\rho t/2-\rho-4+2\beta)} n^{2+(t/2)}\\ &= 0\end{aligned}$$

providing $t > 2\rho^{-1}(\rho + 4 - 2\beta)$. Let $K \uparrow \infty$, $\rho \downarrow \beta$ and $\beta \uparrow \alpha$ to conclude that

$$\dim\big(C_\alpha(X_1)\big) \le \frac{8}{\alpha} - 2.$$

□

## 6. Lower Bounds for Thin Sets.

If $0 \le s < 1$ and $y \in C \equiv C([0,1],\mathbb{R}^d)$ satisfies $y = y^s$ (recall $y^s(\cdot) = y(s \wedge \cdot)$), let $R_s(y,\cdot)$ denote the canonical measure on $M_F(C)$ associated with the infinitely divisible random measure $\mathbb{Q}_{s,\delta_y}(H_1 \in \cdot)$. Let $\mathbb{P}^*(s,y)(\cdot) = R_s(y,\cdot)/R_s\big(y, M_F(C)\big)$be the corresponding cluster law. Then the historical version of (3.9) (see Proposition 3.3 of [DP]) states that under $\mathbb{Q}_{s,\delta_y}$, $H_1$ is equal in law to a Poisson (mean 1/2) superposition of i.i.d. clusters with law $\mathbb{P}^*(s,y)$. It is easy to use the path-valued process and the historical analogue of (3.7) to see that

$$\mathbb{P}^*(0,x)(A) = 2\mathbb{N}_x\left(\int_0^\sigma 1(W_s \in \cdot)d_sL_s^1 \in A, L_\sigma^1 > 0\right). \tag{6.1}$$

We will work with $\mathbb{P}^* = \mathbb{P}^*(0,0)$ for most of this section and write $H$ for the canonical variable on ($M_F(C)$, Borel sets).

If $r_sH(\phi) = \int \phi(y^s)H(dy)$, then for $0 \le s < 1$, $r_sH$ is a.s. purely atomic (Proposition 3.5 of [DP]). Let $H_s^*$ be the random measure which assigns mass one to each point in $S(r_sH)$. This process (under $\mathbb{P}^*$) will play a key role in the main construction in this section and so the non-expert reader may appreciate a more concrete description of $H^*$ in terms of branching particle systems. Set $T_1 = 0$ and let $\{Y_1(t) : t \in [0,1]\}$ be a $d$-dimensional Brownian motion starting at zero, and let $T_2$ be an independent random variable, uniformly distributed on $[0,1]$. Conditional on $\{Y_1, T_2\}$ let $\{Y_2(t) : t \in [0,1]\}$ coincide with $Y_1(t)$ up to $t = T_2$ and then evolve as an independent Brownian motion after $T_2$. Conditional on $\{Y_1, Y_2, T_2\}$ let $T_3$ and $T_4$ be independent random variables uniformly distributed on $[T_2, 1]$ and conditionally on $\{T_2, T_3, T_4, Y_1, Y_2\}$ for $j = 3$ or $4$ let $Y_j$ coincide with $Y_{j-2}$ up to time $T_j$ and then evolve as an independent Brownian motion up to $t = 1$. Moreover $Y_3$ and $Y_4$ are conditionally independent. Continuing in this way, we construct a sequence of branching times $\{T_i\}$ and Brownian paths $\{Y_i(t) : t \in [0,1]\}$ with $Y_i$ branching off some $Y_j$ $(j < i)$ at time $T_i$. Then $H_t^* = \sum_i 1(T_i \le t)\delta_{Y_i^t}$ is equal in law to the process (also denoted by $H_t^*$!) described above. This is the empirical measure of a system of branching Brownian motions with inhomogeneous branching rate $(1-s)^{-1}ds$ and so this equivalence follows from Theorem 3.9(b) of [DP]. Note that $S(H_t^*) = \{Y_i^t : i \in \mathbb{N}\} = \{Y_i^t : T_i \le t\}$.

**Notation.**

If $y \in C, 0 \le s < t$, let $S(y,s,t) = \{w \in C : w^s = y^s, w^t \ne y^t\}$.

$$\mathcal{F}_s^* = \bigcap_{n=1}^\infty \big(\sigma(H_u^* : u \le s + n^{-1}) \vee \{\mathbb{P}^* - \text{ null sets}\}\big).$$

We require the following results from [DP]. Until otherwise indicated we work with respect to $\mathbb{P}^*$.

**Theorem 6.1**. Let $0 \le s < t < 1$.

(a) If $H(y)(A) = H(\{w : w^s = y^s, w \in A\})$, then conditional on $\mathcal{F}_s^*$, $\{H(y) : y \in S(H_s^*)\}$ are independent and $H(y)$ has law $\mathbb{P}^*(s,y)$ $\big(y \in S(H_s^*)\big)$. Moreover

$$\{H(\{w \in C : w^s = y^s\}) : y \in S(H_s^*)\}$$

are conditionally independent exponential random variables with mean $2(1-s)$.

(b) Conditional on $\mathcal{F}_s^*$, $\{\ \#\ \big(S(y,s,t) \cap S(H_t^*)\big) + 1\ :\ y \in S(H_s^*)\}$ are i.i.d. geometric random variables with mean $(1-s)/(1-t)$.

**Remarks**. (a) is an easy consequence of Proposition 3.5(b) of [DP] which gives this result under $\mathbb{Q}_{0,\delta_0}$ instead of $\mathbb{P}^*$. Simply decompose $H$ (in that result) into a Poisson superposition of clusters, note that the number of clusters is specified by the conditioned information and consider the equality when there is only one cluster. The last assertion in (a) is then immediate from Proposition 3.3 (b) of [DP] (recall that, with our branching rate of 4, $\Phi(\lambda) = 2\lambda^2$ in the notation of [DP]).

(b) is immediate from the proof of Theorem 3.11(a) of [DP]. Although the result is stated for $\mathbb{Q}_{0,m}$, the proof proceeds under $\mathbb{P}^*$ and uses the inhomogeneous branching particle system described above.

Fix $\gamma \in (1,2)$. We now construct a sequence of random measures on $C$ which will converge to a random measure $K$ such that $K\big(y(1) \in \cdot\big)$ will be supported by $\gamma$-thin points of a super-Brownian motion at $t = 1$. Recall $h(t) = 5(t \log^+ 1/t)^{1/2}$ ($\log^+ x = |\log x| \vee 1$). We choose a sequence $\{\delta_n, n \in \mathbb{N}\}$ which decreases in $(0,1]$, and satisfies, for some $\overline{\delta} > 0$ and all $n \in \mathbb{N}$:

(6.2$a$) $\quad \delta_{n+1} < h^{-1}\left(\delta_n^{\gamma/2}\right) \equiv \eta_n \quad$ ( set $\delta_0 = \eta_0 \equiv 1$)

(6.2$b$) $\quad$ If $\varepsilon_n = \delta_n \prod_{j=1}^{n-1} \delta_j^2 \eta_j^{-2}$ (set $\varepsilon_1 = \delta_1$), then $\delta_1^{\frac{\gamma}{2}-1} \sum_{k=2}^{\infty} \varepsilon_k \delta_k^{-1} \eta_{k-1} < \overline{\delta}$

and, in particular, $\lim \varepsilon_n = 0$

(6.2$c$) $\quad \delta_n < n^{-3} \delta_{n-1}$

Note that $\eta_n < \delta_n$. It is clearly possible to inductively choose $\{\delta_n\}$ so that the above holds for any given $\overline{\delta} > 0$. For example, in (6.2b) note that for $k \ge 3$,

$$\varepsilon_k \delta_k^{-1} \eta_{k-1} = \left(\prod_1^{k-2} \delta_j^2 \eta_j^{-2}\right) \delta_{k-1}^2 \eta_{k-1}^{-1}$$

and $\eta_{k-1}^{-1} \le c\delta_{k-1}^{-\gamma} (\log^+ 1/\delta_{k-1})^{-1}$ with $\gamma < 2$ and for $k = 2$,

$$\delta_1^{\gamma/2-1} \varepsilon_2 \delta_2^{-1} \eta_1 = \delta_1^{1+\gamma/2} \eta_1^{-1} \le c\delta_1^{1-\gamma/2} (\log^+ 1/\delta_1)^{-1}.$$

Let $K_n$ $(n \in \mathbb{N})$ denote the random measure on $C$ assigning mass $\varepsilon_n$ to each path in

$$\{y \in S(H^*_{1-\delta_n}) : H(S(y, 1-\delta_k, 1-\eta_k)) = 0 \text{ for all } 1 \leq k < n\}.$$

Note these are the paths in $S(H^*_{1-\delta_n})$ which have no cousins branching off in the set of times $I_n = \bigcup_{k=1}^{n-1} [1-\delta_k, 1-\eta_k)$. This will effectively create $\gamma$-thin points by looking at appropriate length scales.

**Notation**. If $y \in C$, let $[y]_n = \{w \in C : w^{1-\delta_n} = y^{1-\delta_n}\}$.

**Proposition 6.2**. (a) $\{(K_n(1), \mathcal{F}^*_{1-\delta_n}) : n \in \mathbb{N}\}$ is a mean one $L^2$-bounded martingale.

(b) There is an a.s. finite random measure $K$ on $C$ such that $K_n \xrightarrow{\text{a.s.}} K$ in $M_F(C)$ and $S(K) \subset S(H)$ $\mathbb{P}^*$-a.s.

**Proof**. Fix $n_0 \in \mathbb{N}$ and $\phi : C \to \mathbb{R}$ bounded continuous such that $\phi(y) = \phi(y^{1-\delta_{n_0}})$. We claim that $\{\big(K_n(\phi), \mathcal{F}^*_{1-\delta_n}\big) : n \geq n_0\}$ is a martingale. Fix $n \geq n_0$ and let $\{U_i : i \in \mathbb{N}\}$ and $\{G_i : i \in \mathbb{N}\}$ be independent collections of iid random variables (defined on an augmented space, if necessary) which are independent of $\mathcal{F}^*_{1-\delta_n}$ and such that $G_i$ is geometric with mean $\eta_n/\delta_{n+1}$ and $\mathbb{P}^*(U_i = 1) = 1 - \mathbb{P}^*(U_i = 0) = \eta_n \delta_n^{-1}$. By Theorem 6.1 (b) for $K_n$-a.a. $y$

$$\begin{aligned}
\mathbb{P}^*\left(K_{n+1}([y]_n) \mid \mathcal{F}^*_{1-\delta_n}\right)(\omega) &= \varepsilon_{n+1} \mathbb{P}^*\left(U_1 G_1\right) \\
&= \varepsilon_{n+1} \eta_n \delta_n^{-1} \eta_n \delta_{n+1}^{-1} \\
&= \varepsilon_n = K_n([y]_n)(\omega),
\end{aligned}$$

and so

$$\begin{aligned}
\mathbb{P}^*\left(K_{n+1}(\phi) \mid \mathcal{F}^*_{1-\delta_n}\right) &= \mathbb{P}^*\left(\sum_{y \in S(K_n)} \phi(y) K_{n+1}([y]_n) \mid \mathcal{F}^*_{1-\delta_n}\right) \\
&= \sum_{y \in S(K_n)} K_n([y]_n)\phi(y) \quad \text{(by the above)} \\
&= K_n(\phi).
\end{aligned}$$

In particular if $\phi \equiv 1$ we see that $\{(K_n(1), \mathcal{F}^*_{1-\delta_n}) : n \in \mathbb{N}\}$ is a martingale, and by Theorem 6.1(b),

$$\mathbb{P}^*\big(K_1(1)\big) = \varepsilon_1 \mathbb{P}^*\left(\# S(H^*_{1-\delta_1})\right) = \varepsilon_1 \delta_1^{-1} = 1.$$

Turning to $L^2$-boundedness, we may use the above notation and reasoning to write

$$\begin{aligned}
\mathbb{P}^*\left(\big(K_{n+1}(1) - K_n(1)\big)^2 \mid \mathcal{F}^*_{1-\delta_n}\right) &= \varepsilon_{n+1}^2 \mathbb{P}^*\left(\left[\sum_i 1\big(i \leq \varepsilon_n^{-1} K_n(1)\big)(U_i G_i - \varepsilon_n \varepsilon_{n+1}^{-1})\right]^2 \mid \mathcal{F}^*_{1-\delta_n}\right) \\
&= \varepsilon_{n+1}^2 \sum_i 1\big(i \leq \varepsilon_n^{-1} K_n(1)\big) \mathbb{P}^*\left(\big(U_i G_i - \eta_n^2 \delta_n^{-1} \delta_{n+1}^{-1}\big)^2 \mid \mathcal{F}^*_{1-\delta_n}\right) \\
&= \varepsilon_{n+1}^2 \varepsilon_n^{-1} K_n(1) \left[\eta_n^2 \delta_n^{-1} \delta_{n+1}^{-1}\right] \left[2\eta_n \delta_{n+1}^{-1} - 1 - \eta_n^2 \delta_n^{-1} \delta_{n+1}^{-1}\right] \\
&= \varepsilon_{n+1} K_n(1) \delta_{n+1}^{-1} \big(2(\eta_n - \delta_{n+1})\big) + K_n(1)\left(\varepsilon_{n+1} - \varepsilon_{n+1} \eta_n^2 \delta_n^{-1} \delta_{n+1}^{-1}\right).
\end{aligned}$$

Take means and use the facts that $\mathbb{P}^*\big(K_n(1)\big) = 1$ and $\mathbb{P}^*\big(K_1(1)^2\big) = 2-\delta_1$ (from Theorem 6.1(b) with $s = 0$ and $t = 1-\delta_1$) to see that

$$\mathbb{P}^*\big(K_{n+1}(1)^2\big) = 2\sum_{k=1}^{n+1} \varepsilon_k \delta_k^{-1}(\eta_{k-1} - \delta_k) + \varepsilon_{n+1} \tag{6.3}$$

and

$$\mathbb{P}^*\left((K_{n+1}(1) - K_1(1))^2\right) = 2\sum_{k=2}^{n+1} \varepsilon_k \delta_k^{-1}(\eta_{k-1} - \delta_k) + \varepsilon_{n+1} - \varepsilon_1. \tag{6.4}$$

By (6.2b) and (6.3) we see that $\{K_n(1)\}$ is an $L^2$-bounded martingale.

Fix a countable determining class $\mathcal{C}$ of bounded continuous functions $\phi$ such that $\phi(y) = \phi(y^t)$ for some $t < 1$ ($t$ may depend on $\phi$). By the martingale convergence theorem and the above ($\|K_n(\phi)\|_1 \le \|\phi\|_\infty \|K_n(1)\|_1 = \|\phi\|_\infty$) $\{K_n(\phi)\}$ converges a.s. for each $\phi$ in $\mathcal{C}$. Since $\bigcup_n S(K_n) \subset \{y^s : s \in [0,1], y \in S(H)\}$ which is a.s. compact (Theorem 8.10 of [DP]) and $\sup_n K_n(1) < \infty$ a.s., we see that $\{K_n\}$ are a.s. tight. Choose $\omega$ outside a null set so that $\{K_n(\omega) : n \in \mathbb{N}\}$ is tight and $\{K_n(\omega)(\phi) : n \in \mathbb{N}\}$ converges for all $\phi$ in $\mathcal{C}$. Then $\{K_n(\omega)\}$ converges in $M_F(C)$. Let $K(\omega)$ denote the a.s. limit.

It remains to show that $S(K) \subset S(H)$ a.s. By definition

$$\bigcup_{n \ge N} S(K_n) \subset \{y^s : 1 - \delta_N \le s \le 1, y \in S(H)\}$$

and this latter set is a.s. compact because $S(H)$ is. The above a.s. convergence implies

$$S(K) \subset \bigcap_{N=1}^{\infty} \{y^s : 1 - \delta_N \le s \le 1, y \in S(H)\} \text{ a.s.}$$

The a.s. compactness of $S(H)$ easily shows that the set on the right is $S(H)$ a.s. and the proof is complete. $\square$

**Corollary 6.3**. For any $\varepsilon_0 > 0$ we may choose $\overline{\delta} > 0$ in (6.2b) so that $\mathbb{P}^*\big(K(1) > 0\big) \ge 1 - \varepsilon_0$.

**Proof**. Since $\{K_n(1)\}$ is an $L^2$-bounded martingale we may let $n \to \infty$ in (6.4) to conclude

$$\mathbb{P}^*\left(\big(K(1) - K_1(1)\big)^2\right) = 2\sum_{k=2}^{\infty} \varepsilon_k \delta_k^{-1}(\eta_{k-1} - \delta_k) - \varepsilon_1 \le 2\overline{\delta}\delta_1^{1-\gamma/2}.$$

Therefore if $q = (1-\gamma/2)/2 (> 0)$, we have

$$\begin{aligned}
\mathbb{P}^*\big(K(1) \le \delta_1^q\big) &\le \mathbb{P}^*\big(K_1(1) \le 2\delta_1^q\big) + \mathbb{P}^*\big(|K(1) - K_1(1)| > \delta_1^q\big) \\
&\le 1 - (1-\delta_1)^{[2\delta_1^q]} + \delta_1^{-2q} 2\overline{\delta}\delta_1^{1-\gamma/2} \\
&< \varepsilon_0
\end{aligned}$$

provided $\overline{\delta}$ (and hence $\delta_1$ because $\delta_1^{1-\gamma/2}\log^+ 1/\delta_1 < c\overline{\delta}$) is sufficiently small. $\square$

For $y \in C$ let $\mathbb{P}^{(y)}$ denote a law on an appropriate probability space under which $\mathcal{N}(dr, d\nu)$ is a Poisson point process on $[0,1] \times M_F(C)$ with intensity $2\mathbb{P}^*(r,y)(d\nu)(1-r)^{-1}dr$. Recall that $P_0$ is Wiener measure on $C$ starting at 0. If $\psi : C \times M_F(C) \to \mathbb{R}$ is bounded and measurable then the representation of the Palm measure for the canonical measure, in [DP] (see Proposition 4.1.5) implies that

$$\mathbb{P}^*\Big(\int \psi(y,H)H(dy)\Big) = 2\iint \psi\big(y, \int_0^1\int \nu\mathcal{N}(dr,d\nu)\big)d\mathbb{P}^{(y)}dP_0(y). \tag{6.5}$$

Note that in (6.5) $\nu$ denotes a cluster which branches off $y$ at time $r$. More precisely Proposition 4.1.8 of [DP] implies that if $H = \iint \nu\mathcal{N}(dr,d\nu)$ then

$$H\big(S(y,s,t)\cap A\big) = \int_s^t\int \nu(A)\mathcal{N}(dr,d\nu) \quad \mathbb{P}^{(y)}\text{-a.s.} \quad \text{for } 0 \le s < t \le 1 \text{ and } y \in C. \tag{6.6}$$

Let $\tilde{I}_n = \bigcup_{k=0}^{n-1}[1-\eta_k, 1-\delta_{k+1})$, $\tilde{I} = \tilde{I}_\infty$, and set $S(y,j) = S(y, 1-\delta_j, 1-\eta_j)$. From the vantage of a typical trajectory $y$ chosen according to $K$, $H$ only has clusters which branch off from $y$ at times in $\tilde{I}$ and so the following analogue of (6.5) for $K$ is to be expected.

**Proposition 6.4**. If $\phi : C \times M_F(C) \to \mathbb{R}$ is bounded and Borel measurable, then

$$\mathbb{P}^*\Big(\int \phi(y,H)K(dy)\Big) = \int\int \phi\Big(y, \int_{\tilde{I}}\int \nu\mathcal{N}(dr,d\nu)\Big)\,d\mathbb{P}^{(y)}dP_0(y).$$

**Proof**. If $H^{n,y}(A) = H(A \cap [y]_n^c)$ then Theorem 6.1 shows that, as $n \to \infty$, $H^{n,y} \to H$ in total variation in probability, uniformly in $y$. We may assume $\phi$ is uniformly continuous (use the total variation norm on $M_F(C)$ or any metric inducing the weak topology bounded by this norm). It is then easy to check that

$$\begin{aligned}\mathbb{P}^*\Big(\int \phi(y,H)K(dy)\Big) &= \lim_{n\to\infty}\mathbb{P}^*\Big(\int \phi(y,H^{n,y})K_n(dy)\Big)\\ &= \lim_{n\to\infty}\varepsilon_n\mathbb{P}^*\Big(\int \phi\Big(y^{1-\delta_n}, H^{n,y}\Big)1\Big(H\big(S(y,j)\big)=0 \quad \forall j<n\Big)H([y]_n)^{-1}H(dy)\Big)\end{aligned}$$

$$= \lim_{n\to\infty} \varepsilon_n 2 \iint \phi\Big(y^{1-\delta_n}, \int_0^{1-\delta_n}\int \nu\mathcal{N}(dr,d\nu)\Big) 1\left(\int_{I_n}\int \nu(1)\mathcal{N}(dr,d\nu)=0\right)$$

$$\left(\int_{1-\delta_n}^{1}\int \nu(1)\mathcal{N}(dr,d\nu)\right)^{-1} d\mathbb{P}^{(y)}\, dP_0(y)$$

(by (6.5) and (6.6))

$$= \lim_{n\to\infty} \varepsilon_n 2 \iint \phi\Big(y^{1-\delta_n}, \int_{\tilde{I}_n}\int \nu\mathcal{N}(dr,d\nu)\Big) 1\left(\int_{I_n}\int \nu(1)\mathcal{N}(dr,d\nu)=0\right)$$

$$\left(\int_{1-\delta_n}^{1}\int \nu(1)\mathcal{N}(dr,d\nu)\right)^{-1} d\mathbb{P}^{(y)}\, dP_0(y)$$

$$= 2\lim_{n\to\infty} \int \left[\int \phi\Big(y^{1-\delta_n}, \int_{\tilde{I}_n} \nu\mathcal{N}(dr,d\nu)\Big) d\mathbb{P}^{(y)}\right] \Bigg[\varepsilon_n 1\left(\int_{I_n}\int \nu(1)\mathcal{N}(dr,d\nu)=0\right)$$

$$\left(\int_{1-\delta_n}^{1}\int \nu(1)\mathcal{N}(dr,d\nu)\right)^{-1} d\mathbb{P}^{(y)}\Bigg]\, dP_0(y).$$

The second term in square brackets is independent of $y$ because $\mathbb{P}^*(r,y)(\nu(1)\in\cdot)$ is (see (3.7) of [DP]). Set $\phi\equiv 1$ in the above to see this term approaches $1/2$ as $n\to\infty$ and hence conclude that

$$\mathbb{P}^*\Big(\int \phi(y,H)K(dy)\Big) = \iint \phi\Big(y, \int_{\tilde{I}}\int \nu\mathcal{N}(dr,d\nu)\Big) d\mathbb{P}^{(y)}\, dP_0(y). \ \square$$

We now obtain a second moment result which will yield the key estimate in this section. If $y=y^s$ then $P_{s,y}$ is the law on $C$ of Brownian motion which starts at time $s$ with past history $y$ (see Section 2.2. of [DP]).

**Theorem 6.5**. If $\phi:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ is bounded and Borel measurable, then

$$\mathbb{P}^*\Big(\iint \phi\big(y_1(1),y_2(1)\big)K(dy_1)K(dy_2)\Big) = 2\sum_{k=1}^{\infty}\varepsilon_k\delta_k^{-1}\int_{1-\eta_{k-1}}^{1-\delta_k}\Big[\iint \phi\big(y_1(1),y_2(1)\big)P_{r,y_1^r}(dy_2)$$

$$P_0(dy_1)\Big]\, dr \tag{6.7}$$

$$< \infty.$$

**Remark**. It is not hard to use branching particle systems to convince oneself of this formula. Note that $r \in \tilde{I}$ is the time at which two paths chosen according to $K$ branch apart.

**Proof**. It suffices to consider $\phi(y_1, y_2) = \phi_1(y_1)\phi_2(y_2)$ where each $\phi_i$ is bounded and continuous. By polarization we may take $\phi_1 = \phi_2 = f$. Let

$$\tilde{K}_n(\psi) = \varepsilon_n(2\delta_n)^{-1} \int \psi(y^{1-\delta_n}) 1\Big(H\big(S(y,j)\big) = 0 \quad \forall j < n\Big) H(dy).$$

Then, identifying $f$ with the map $y \to f\big(y(1)\big)$ on $C$, we have

$$\begin{aligned}
\mathbb{P}^*\left(\big(K_n(f) - \tilde{K}_n(f)\big)^2\right) &= \mathbb{P}^*\left(\varepsilon_n^2 \left[\sum_{y \in S(H^*_{1-\delta_n})} f\big(y(1-\delta_n)\big) 1\Big(H\big(S(y,j)\big) = 0 \quad \forall j < n\Big) \left(\frac{H([y]_n)}{2\delta_n} - 1\right)\right]^2\right) \\
&= \mathbb{P}^*\left(\varepsilon_n^2 \sum_{y \in S(H^*_{1-\delta_n})} f\big(y(1-\delta_n)\big)^2 1\Big(H\big(S(y,j)\big) = 0 \quad \forall j < n\Big) \right. \\
&\qquad \left. \times \mathbb{P}^*\left(\left(\frac{H([y]_n)}{2\delta_n} - 1\right)^2 \mid \mathcal{F}^*_{1-\delta_n}\right)\right) \quad \text{(Theorem 6.1(a))} \\
&\le \|f\|_\infty^2 \varepsilon_n \mathbb{P}^*\big(K_n(1)\big) = \varepsilon_n \|f\|_\infty^2 \to 0 \text{ as } n \to \infty.
\end{aligned}$$

It therefore suffices to show (recall $\sup_n K_n(1) \in L^2$ by Proposition 6.2)

(6.8)
$$\lim_{n\to\infty} \mathbb{P}^*\big(\tilde{K}_n(f)K(f)\big) \text{ equals the right-hand side of (6.7) with } \phi(x_1, x_2) = f(x_1)f(x_2).$$

Apply Proposition 6.4 with

$$\phi(y, H) = f\big(y(1)\big)\varepsilon_n(2\delta_n)^{-1} \int f\big(y_2(1-\delta_n)\big) 1\Big(H\big(S(y_2,j)\big) = 0 \quad \forall j < n\Big) H(dy_2)$$

to see that

$$\begin{aligned}
\mathbb{P}^*\big(\tilde{K}_n(f)K(f)\big) = \varepsilon_n(2\delta_n)^{-1} \int f\big(y_1(1)\big) \iiiint_{\tilde{I}} f\big(y_2(1-\delta_n)\big) \\
1\left(\int_{\tilde{I}} \int \nu'\big(S(y_2,j)\big) d\mathcal{N}(r', \nu') = 0 \quad \forall j < n\right) \\
\nu(dy_2)\mathcal{N}(dr, d\nu) d\mathbb{P}^{(y_1)} dP_0(y_1).
\end{aligned}$$

For $\mathcal{N}$-a.a. $(r,\nu)$ in $[1-\eta_k, 1-\delta_{k+1}) \times M_F(C)$ and $\nu$-a.a. $y_2$, $y_2^r = y_1^r$ and so $S(y_2,j) = S(y_1,j)$ for all $j \le k$. Use (6.6) to see that for $\mathcal{N}$-a.a. $(r,\nu)$ in $(1-\eta_k, 1-\delta_{k+1}) \times M_F(C)$ and $\nu$-a.a. $y_2$ and all $j \le k$

$$\int\limits_{\tilde I}\int \nu'\big(S(y_2,j)\big)d\mathcal{N}(r',\nu') = \int\limits_{\tilde I\cap[1-\delta_j,1-\eta_j)}\int \nu'(1)d\mathcal{N}(r',\nu') = 0.$$

Therefore

$$\begin{aligned}\mathbb{P}^*\big(\tilde K_n(f)K(f)\big) = \varepsilon_n(2\delta_n)^{-1}\sum_{k=0}^{\infty}\int f\big(y_1(1)\big)\int\int_{1-\eta_k}^{1-\delta_{k+1}}\iint f\big(y_2(1-\delta_n)\big)\\ 1\left(\int\limits_{I}\int \nu'\big(S(y_2,j)\big)d\mathcal{N}(r',\nu') = 0 \quad \forall\, k<j<n\right)\\ \nu(dy_2)\mathcal{N}(dr,d\nu)d\mathbb{P}^{(y_1)}dP_0(y_1).\end{aligned}$$

Fix $0 \le k < j < n$. We claim that $\mathbb{P}^{(y_1)}$-a.s.

$$\begin{aligned}\int\limits_{\tilde I}\int \nu'\big(S(y_2,j)\big)d\mathcal{N}(r',\nu') = \nu\big(S(y_2,j)\big) \quad \nu\text{-a.a. } y_2 \quad \mathcal{N}\text{-a.a. } (r,\nu)\\ \text{in } [1-\eta_k, 1-\delta_{k+1}) \times M_F(C).\end{aligned} \tag{6.9}$$

The left side is clearly a.s. greater than or equal to the right side as $(r,\nu)$ is a point contributing to the left side. Note that

$$\begin{aligned}&\mathbb{P}^{(y_1)}\left(\int_0^1\int \nu\left(\{w : w^{r'} \ne y_1^{r'} \text{ or } w^s = y_1^s \text{ for some } s > r'\}\right)\mathcal{N}(dr',d\nu')\right)\\ &= 2\int_0^1\int \nu\left(\{w : w^{r'} \ne y_1^{r'} \text{ or } w^s = y_1^s \text{ for some } s > r'\}\right)\mathbb{P}^*(r',y_1)(d\nu)(1-r')^{-1}dr'\\ &= \int_0^1 (1-r')^{-1}P_{r',y_1^{r'}}\left(\{w : w^{r'} \ne y_1^{r'} \text{ or } w^s = y_1^s \text{ for some } s > r'\}\right)dr'\\ &= 0.\end{aligned}$$

This shows that

$$w^{r'} = y_1^{r'} \text{ and } w^s \ne y_1^s \text{ for all } s > r' \text{ for } \nu'\text{-a.a. } w \text{ for } \mathcal{N}\text{-a.a. } (r',\nu') \quad \mathbb{P}^{(y_1)}\text{-a.s.} \tag{6.10}$$

Fix $y_2 \in C$ and $r$ in $[1-\eta_k, 1-\delta_{k+1})$ such that $y_2^r = y_1^r$ but $y_2^s \neq y_1^s$ for all $s > r$ ($(r, y_2)$ will be chosen as in (6.9)). Then (recall $j > k$)

$$\begin{aligned}
\int\limits_{(0,r)} \int \nu'\big(S(y_2,j)\big)d\mathcal{N}(r',\nu') &\leq \int\limits_{(0,r)} \int \nu'\big(\{w : w^{1-\delta_j} = y_2^{1-\delta_j}\}\big)d\mathcal{N}(r',\nu') \\
&\leq \int\limits_{(0,r)} \int \nu'\big(\{w : w^r = y_1^r\}\big)d\mathcal{N}(r',\nu') \\
&= 0 \quad \text{(by (6.10))}
\end{aligned}$$

and

$$\begin{aligned}
\int\limits_{(r,1]} \int \nu'\big(S(y_2,j)\big)d\mathcal{N}(r',\nu') &\leq \int\limits_{(r,1]} \int \nu'\big(\{w : w^{1-\delta_j} = y_2^{1-\delta_j}\}\big)d\mathcal{N}(r',\nu') \\
&\leq \int\limits_{(r,1]} \int \nu'\big(\{w : w^{r'} \neq y_1^{r'}\}\big)d\mathcal{N}(r',\nu') \\
&= 0 \quad \text{(by (6.10))}.
\end{aligned}$$

This shows that $\mathbb{P}^{(y_1)}$-a.s. for $\mathcal{N}$-a.s. $(r,\nu)$ in $[1-\eta_k, 1-\delta_{k+1}) \times M_F(C)$ and $\nu$-a.a. $y_2$,

$$\int\limits_{\tilde{I}} \int \nu'\big(S(y_2,j)\big)d\mathcal{N}(r',\nu') = \int\limits_{\{r\}} \int \nu'\big(S(y_2,j)\big)d\mathcal{N}(r',\nu') = \nu\big(S(y_2,j)\big),$$

where the last holds because $\mathcal{N}\big(\{r\} \times M_F(C)\big) \leq 1 \;\forall\, r \geq 0$ a.s. This proves (6.9) and so

$$\begin{aligned}
\mathbb{P}^*\big(\tilde{K}_n(f)K(f)\big) &= \varepsilon_n(2\delta_n)^{-1}\sum_{k=0}^{\infty}\int f\big(y_1(1)\big)\int \int\limits_{1-\eta_k}^{1-\delta_{k+1}} \iint f\big(y_2(1-\delta_n)\big) \\
&\qquad 1\Big(\nu\big(S(y_2,j)\big) = 0 \,\forall\, k < j < n\Big)\nu(dy_2)\mathcal{N}(dr,d\nu)d\mathbb{P}^{(y_1)}dP_0(y_1) \\
&= 2\varepsilon_n(2\delta_n)^{-1}\sum_{k=0}^{\infty}\int f\big(y_1(1)\big) \int\limits_{1-\eta_k}^{1-\delta_{k+1}} \iint f\big(y_2(1-\delta_n)\big) \\
&\qquad 1\Big(H\big(S(y_2,j)\big) = 0 \,\forall\, k < j < n\Big)H(dy_2)d\mathbb{P}^*(r,y_1)(1-r)^{-1}dr P_0(dy_1).
\end{aligned}$$

If under $\mathbb{P}^{(r,y_1)}$, $\mathcal{N}(du,d\nu)$ is a Poisson point process on $[0,1] \times M_F(C)$ with intensity $1(r \leq u < 1)\; 2(1-u)^{-1}d\mathbb{P}^*(u,y_1)\,du$, then the analogue of (6.5) for $\mathbb{P}^*(r,y_1)$ (see Proposition 4.1.5 of [DP]) shows the above equals

$$\varepsilon_n\delta_n^{-1}\sum_{k=0}^{\infty}\int f\big(y_1(1)\big)\int\limits_{1-\eta_k}^{1-\delta_{k+1}} 2\iint f\big(y_2(1-\delta_n)\big)$$

$$1\left(\int\limits_r^1\int \nu\big(S(y_2,j)\big)d\mathcal{N}(u,\nu)=0 \quad \text{for } k<j<n\right) d\mathbb{P}^{(r,y_2)}P_{r,y_1^r}(dy_2)\,dr\,P_0(dy_1)$$

$$=\varepsilon_n\delta_n^{-1}\sum_{k=0}^{\infty}\int f\big(y_1(1)\big)\int\limits_{1-\eta_k}^{1-\delta_{k+1}}\Bigg[2\iint f\big(y_2(1-\delta_n)\big)$$

$$1\left(\int\limits_{1-\delta_j}^{1-\eta_j}\int \nu(1)d\mathcal{N}(u,\nu)=0 \quad \text{for } k<j<n\right) d\mathbb{P}^{(r,y_2)}P_{r,y_1^r}(dy_2)\Bigg]\;drP_0(dy_1)\ \text{(as for 6.6))}$$

$$=2\varepsilon_n\delta_n^{-1}\sum_{k=0}^{\infty}\exp\left\{-2\sum_{k<j<n}\int\limits_{1-\delta_j}^{1-\eta_j}(1-u)^{-1}\,du\right\}\int\limits_{1-\eta_k}^{1-\delta_{k+1}}\iint f\big(y_1(1)\big)f\big(y_2(1-\delta_n)\big)$$

$$P_{r,y_1^r}(dy_2)P_0(dy_1)\,dr$$

$$=2\sum_{k=0}^{\infty}\left(\prod_{1\le j\le k\wedge(n-1)}\delta_j^2\eta_j^{-2}\right)\int\limits_{1-\eta_k}^{1-\delta_{k+1}}\iint f\big(y_1(1)\big)f\big(y_2(1-\delta_n)\big)P_{r,y_1^r}(dy_2)P_0(dy_1)\,dr$$

$$=2\sum_{k=1}^{\infty}\big(\varepsilon_{k\wedge n}\delta_{k\wedge n}^{-1}\big)\int\limits_{1-\eta_{k-1}}^{1-\delta_k}\left[\iint f\big(y_1(1)\big)f\big(y_2(1-\delta_n)\big)P_{r,y_1^r}(dy_2)P_0(dy_1)\right]\,dr.$$

Use (6.2b) and Dominated Convergence to see that as $n\to\infty$ the above expression converges to the righthand side of (6.7) with $\phi(x_1,x_2)=f(x_1)f(x_2)$. This establishes (6.8) and the proof is complete. □

Define random measures on $\mathbb{R}^d$ by $X(A)=H\big(\{y: y(1)\in A\}\big)$ and $Y(A)=K\big(\{y: y(1)\in A\}\big)$. If $\varepsilon>0$ let $\psi_\varepsilon(r)=r^{-(4/\gamma-2)}(\log^+\frac{1}{r})^{-(2/\gamma+\varepsilon)}$. We have yet to use the fact that we are working in 3 or more dimensions in this Section. This is needed for the following result.

**Theorem 6.6**. (a) $Y$ is supported by the $\gamma$-thin points in $S(X)$ $\mathbb{P}^*$-a.s. More precisely, $\mathbb{P}^*$-a.s. $S(Y)\subset S(X)$ and

$$\Big(B(x,\delta_n^{1/2})-\overline{B(x,\delta_n^{\gamma/2})}\Big)\cap S(X)=\phi \quad \text{for infinitely many } n \quad Y\text{-a.a. } x.$$

(b) If $\varepsilon>0$ and in addition to (6.2a-c) we have

$$(6.2d)\qquad \sum_{k=1}^{\infty}\varepsilon_k\delta_k^{-1}(\log^+1/\delta_k)^{-\varepsilon}<\infty,$$

then

$$\mathbb{P}^*\left(\iint \psi_\varepsilon(|x_1-x_2|)Y(dx_1)Y(dx_2)\right)<\infty.$$

In particular, the $\gamma$-thin points of $X$ have positive $\psi_\varepsilon$-capacity $\mathbb{P}^*$-a.s.

**Remark**. Note that $\varepsilon_k\delta_k^{-1}\left(\log^+ 1/\delta_k\right)^{-\varepsilon}=\left(\prod_1^{k-1}\delta_j^2\eta_j^{-2}\right)\left(\log^+ 1/\delta_k\right)^{-\varepsilon}$ and so clearly one may inductively define $\{\delta_k\}$ satisfying (6.2 a-d) and hence a random measure $Y$ as above.

**Proof**. (a) $S(Y)\subset S(X)$ a.s. is immediate from $S(K)\subset S(H)$ a.s. The fact that $Y$ is supported by $\gamma$-thin points of $X$ is a simple consequence of Proposition 6.4 and the Lévy modulus of continuity for paths in $S(H)$, as we now show. By Proposition 6.4 it suffices to show that

$$\int_{\tilde I}\int \nu\left(\{w:w(1)\in B\big(y(1),\delta_n^{1/2}\big)-\overline{B\big(y(1),\delta_n^{\gamma/2}\big)}\}\right)\mathcal{N}(dr,d\nu)=0 \tag{6.11}$$

for infinitely many $n$ $\mathbb{P}^{(y)}$-a.s. for $P_0$-a.a. $y$.

First argue as in Lemma 5.1 of [DP] to see that for large enough $n$

$$\begin{aligned}
&\mathbb{P}^{(y)}\left(\int_0^{1-\delta_{n-1}}\int 1_{\tilde I}(r)\nu\Big(\{w:w(1)\in B\big(y(1),\delta_n^{1/2}\big)\}\Big)d\mathcal{N}(r,\nu)>0\right)\\
&\le \mathbb{P}^{(y)}\left(\int_0^{1-\delta_{n-1}}\int \nu\Big(\{w:w(1)\in B\big(y(1),\delta_n^{1/2}\big)\}\Big)d\mathcal{N}(r,\nu)>0\right)\\
&\le c(d)\int_0^{1-\delta_{n-1}}\delta_n^{(d-2)/2}(1-u)^{-d/2}du \quad \text{(as in the derivation of (5.7) of [DP])}\\
&\le c(d)(\delta_n/\delta_{n-1})^{(d-2)/2}.
\end{aligned}$$

This is summable by (6.2c) and so

$$\int_0^{1-\delta_{n-1}}\int 1_{\tilde I}(r)\nu\Big(\{w:w(1)\in B\big(y(1),\delta_n^{1/2}\big)\}\Big)d\mathcal{N}(r,\nu)=0 \quad \text{for large } n \quad \mathbb{P}^{(y)}\text{-a.s.} \quad \forall\, y\in C. \tag{6.12}$$

By Lemma 4.3 of Le Gall et al (1995) there is a $p=p(d)>0$ such that $\forall\, y\in C$

$$\begin{aligned}
&\mathbb{P}^{(y)}\left(\int_{1-\delta_{n-1}}^{1-\eta_n}\int 1_{\tilde I}(r)\nu\Big(\{w:w(1)\in B\big(y(1),\delta_n^{1/2}\big)\}\Big)d\mathcal{N}(r,\nu)=0\right)\\
&\ge \mathbb{P}^{(y)}\left(\int_0^{1-\delta_n}\int \nu\Big(\{w:w(1)\in B\big(y(1),\delta_n^{1/2}\big)\}\Big)d\mathcal{N}(r,\nu)=0\right)\ge p.
\end{aligned}$$

(We have used the connection between the excursion measures in Le Gall et al (1995) and the canonical measure of the historical process–see (6.1).) The Borel-Cantelli Lemma now implies (note that the upper limit of integration in the following may be replaced by $1-\delta_n$)

$$(6.13)\qquad \int_{1-\delta_{n-1}}^{1-\eta_n}\int 1_{\tilde I}(r)\nu\Big(\{w : w(1)\in B\big(y(1),\delta_n^{1/2}\big)\}\Big)d\mathcal{N}(r,\nu)=0 \quad \text{infinitely often}$$

$$\mathbb{P}^{(y)}\text{-a.s.}\quad P_0\text{-a.a. } y.$$

If $H=\int_0^1\int \nu\mathcal{N}(dr,d\nu)$ then by the uniform Lévy modulus (Theorem 8.7 of [DP]) w.p.1 there is a $\delta(H)>0$ such that for all $w$ in $S(H)$ and all $s,\,t\in[0,1]$ with $|t-s|<\delta(H)$, $|w(t)-w(s)|\le\frac{1}{2}h(|t-s|)$. By (6.6) we have for $P_0$-a.a. $y$ and $\mathbb{P}^{(y)}$-a.s.

$$(6.14)\qquad \begin{aligned}&\int_{1-\eta_n}^{1}\int \nu\big(\{w : |w(1)-y(1)|\ge\delta_n^{\gamma/2}\}\big)d\mathcal{N}(r,\nu)\\ &=H\big(\{w : |w(1)-y(1)|\ge\delta_n^{\gamma/2},\, w^{1-\eta_n}=y^{1-\eta_n}\}\big)\\ &=0\quad \text{for } n \text{ sufficiently large.}\end{aligned}$$

We have used the above modulus of continuity and Lévy's modulus for $y$ in the last line. (6.12), (6.13) and (6.14) imply (6.11) and hence (a) is proved.

(b) Let $\beta=(2/\gamma+\varepsilon)$, $\alpha=4\gamma^{-1}-2$ and set $\phi(x_1,x_2)=\psi_\varepsilon(|x_1-x_2|)$ in (6.7) to conclude that

$$\begin{aligned}&\mathbb{P}^*\left(\iint\psi_\varepsilon(|x_1-x_2|)Y(dx_1)Y(dx_2)\right)\\ &=2\sum_{k=1}^{\infty}\varepsilon_k\delta_k^{-1}\int_{1-\eta_{k-1}}^{1-\delta_k}P_0\left(\psi_\varepsilon\left(|\,B\big(2(1-r)\big)|\right)\right)\,dr\\ &\le C\sum_{k=1}^{\infty}\varepsilon_k\delta_k^{-1}\int_{\delta_k}^{\eta_{k-1}}u^{-\alpha/2}(\log^+1/u)^{-\beta}\,du\\ &\le C\sum_{k=1}^{\infty}\left(\prod_1^{k-2}\delta_j^2\eta_j^{-2}\right)\eta_{k-1}^{-1-\alpha/2}\delta_{k-1}^2\left(\log^+1/\eta_{k-1}\right)^{-\beta}\\ &\le C\sum_{k=0}^{\infty}\left(\varepsilon_k\delta_k^{-1}\right)\delta_k^{-\gamma(1+\alpha/2)}\left(\log 1/\delta_k\right)^{1+\alpha/2}\delta_k^2\left(\log^+1/\delta_k\right)^{-\beta}\\ &\le C\sum_{k=0}^{\infty}\left(\varepsilon_k\delta_k^{-1}\right)\left(\log^+(1/\delta_k)\right)^{-\varepsilon}.\end{aligned}$$

The finite energy condition for $Y$ is therefore immediate from (6.2d).

The $\gamma$-thin points therefore have positive $\psi_\varepsilon$-capacity on $\{Y \neq 0\}$ a.s. and Corollary 6.3 shows this latter set can have $\mathbb{P}^*$-probability as close to one as we like by choosing $\{\delta_j\}$ appropriately. $\square$

**Remark 6.7**. If $\gamma = 2$ one can no longer choose $\{\delta_n\}$ so that (6.2b) holds and the construction of $K$ and the above results become invalid. One may, however, obtain a version of the above results with $\gamma = 2$ and appropriate logarithmic corrections. More precisely if $\varepsilon_0 > 0$, $p > 1/2$ and $g(\delta) = \delta^2(\log^+ 1/\delta)^p$ one may construct a random measure $Y$ such that $\mathbb{P}^*$-a.s. $S(Y) \subset S(X)$, for some sequence $\delta_n \downarrow 0$,

$$(6.15) \qquad \Big(B\big(x, \delta_n^{1/2}\big) - \overline{B\big(x, g(\delta_n^{1/2})\big)}\Big) \cap S(X) = \phi \quad \text{for infinitely many } n \quad Y--\text{a.a. } x$$

and $\mathbb{P}^*(Y \neq 0) \geq 1 - \varepsilon_0$. Moreover if $\tilde{\psi}_\varepsilon(r) = (\log^+ 1/r)^{2p-1-\varepsilon}$ $(0 < \varepsilon < 2p-1)$, then

$$\mathbb{P}^*\left(\iint \tilde{\psi}_\varepsilon(|x_1 - x_2|) Y(dx_1) Y(dx_2)\right) < \infty.$$

To see this set $\eta_k = h^{-1}\big(f(\delta_k)\big)$ where $f(\delta) = \delta(\log 1/\sqrt{\delta})^p$ and make minor changes in the above arguments. For example, in Corollary 6.3 replace $\delta_1^q$ with $(\log^+ 1/\delta_1)^{\frac{1}{4}-\frac{p}{2}}$.

**Notation**. $\phi_{\varepsilon,\gamma}(r) = r^{\frac{4}{\gamma}-2}(\log^+ 1/r)^{\frac{2}{\gamma}+\varepsilon} (= \psi_\varepsilon(r)^{-1})$.

Here is the main result of this Section. Recall that $X_t$ is super-Brownian motion starting at $m_0$ under its canonical law $\mathbb{P}_{m_0}$, and that $S(X_t)_\gamma$ is the set of $\gamma$-thin points in $S(X_t)$. Recall also that the spatial dimension $d$ is greater than 2.

**Theorem 6.8**. (a) For each $\varepsilon$, $t > 0$ and $m_0 \in M_F(\mathbb{R}^d)$,

$$\phi_{\varepsilon,\gamma} - m\big(S(X_t)_\gamma\big) > 0 \quad \mathbb{P}_{m_0}\text{-a.s. on } \{X_t \neq 0\} \quad \forall\, \gamma \in (1,2).$$

(b)
$$\dim\big(S(X_t)_{\alpha/2}\big) = \dim\big(B_\alpha(X_t)\big) = \dim\big(C_\alpha(X_t)\big) = \frac{8}{\alpha} - 2 \quad \mathbb{P}_{m_0}\text{-a.s.}$$
$$\text{on } \{X_t \neq 0\} \quad \forall \alpha \in [2,4].$$

(c)
$$B_\alpha(X_t) \neq \phi \text{ and } C_\alpha(X_t) \neq \phi \quad \mathbb{P}_{m_0}\text{-a.s. on } \{X_t \neq 0\} \quad \text{if } \alpha \in [2,4].$$

$$S(X_t)_{\alpha/2} = B_\alpha(X_t) = C_\alpha(X_t) = \phi \quad \mathbb{P}_{m_0}\text{-a.s.} \qquad \text{if } \alpha > 4.$$
$$B_\alpha(X_t) = \phi \text{ and } C_\alpha(X_t) = S(X_t) \quad \mathbb{P}_{m_0}\text{-a.s.} \qquad \text{if } \alpha < 2.$$

**Proof**. By Evans-Perkins (1991) we may assume without loss of generality that $m_0 = \delta_0$ and $t = 1$.

(a) The definition of the cluster law $\mathbb{P}^*$ (see Proposition 3.3 of [DP] or recall (3.9) and (6.1)) shows that under $\mathbb{P}_{\delta_0}$, $X_1$ is equal in law to $\sum_{i=1}^{N} X^{(i)}$, where $\{X^{(i)} : i \in \mathbb{N}\}$ are i.i.d. random measures with law $\mathbb{P}^*(X \in \cdot)$ and $N$ is an independent Poisson random variable with mean $\frac{1}{2}$. Theorem 6.6 and the well-known relationship between capacity and Hausdorff measure (e.g. Taylor (1961)) show that if $\varepsilon_0 > 0$ there is a random measure $Y^{(1)}$ supported by $S(X^{(1)})_\gamma$ such that $\mathbb{P}(Y^{(1)} \neq 0) > 1 - \varepsilon_0$ and $Y^{(1)}(A) > 0$ implies $\phi_{\varepsilon,\gamma} - m(A) > 0$ for any analytic $A$ a.s. We may, and shall, choose $Y^{(1)}$ independent of $\{X^{(i)} : i > 1\}$ and $N$. Since $\bigcup_{i=2}^{N} S(X^{(i)})$ is a.s. Lebesgue null, Proposition 6.4 and a Fubini argument show that $Y^{(1)}\left(\bigcup_{i=2}^{N} S(X^{(i)})\right) = 0$ a.s. This implies that $Y^{(1)}$ is supported by $S(X^{(1)})_\gamma \cap \left(\bigcup_{i=2}^{N} S(X^{(i)})\right)^c \subset S(X_1)_\gamma$ where we have set $X_1 = \sum_{i=1}^{N} X^{(i)}$. This shows that except for a set of probability $\varepsilon_0$, $\phi_{\varepsilon,\gamma} - m\big(S(X_1)_\gamma\big) > 0$ on $\{N > 0\} = \{X_1 \neq 0\}$. The result follows.

(b) If $\alpha = 2$ this result is immediate from (1.2) and (1.3), and if $\alpha = 4$ it is clear from Corollary 4.4 and Proposition 5.1. Now fix $\alpha \in (2, 4)$. Corollary 4.4, Proposition 5.1 and (a) show the result for $C_\alpha(X_1)$ and $S(X_1)_{\alpha/2}$. In fact these results imply $\phi_{1,\alpha/2} - m\big(C_\alpha(X_1)\big) > 0$ a.s. on $\{X_1 \neq 0\}$ and $\phi_{1,\alpha/2} - m\big(C_{\alpha'}(X_1)\big) = 0\ \forall\, \alpha' > \alpha$ a.s. This shows that if $4 > \alpha_n > \alpha$ and $\alpha_n \downarrow \alpha$, then

$$\phi_{1,\alpha/2} - m\big(B_\alpha(X_1)\big) = \phi_{1,\alpha/2}\left(C_\alpha(X_1) - \big(\bigcup_n C_{\alpha_n}(X_1)\big)\right) > 0 \text{ a.s. on } \{X_1 \neq 0\}.$$

The result for $B_\alpha(X_1)$ is now clear.

(c) Corollary 4.4 and Proposition 5.1 imply the second assertion and Theorem 4.1 gives the third assertion. By (b) we only need consider $\alpha = 4$ in the first assertion. Since $C_\alpha(X_1) = \phi$ for $\alpha > 4$ a.s. we see that $B_4(X_1) = C_4(X_1)$ a.s. and so we consider only $C_4(X_1)$. By arguing as in the proof of Lemma 2.8 and Corollary 4.4, it suffices to prove that for some $p > 0$, a.s. on $\{X_1 \neq 0\}$ for some $x \in S(X_1)$

$$\text{(6.16)} \qquad \Big(B(x, r_i) - \overline{B\big(x, r_i^2(\log^+ 1/r_i)^p\big)}\Big) \cap S(X_1) = \phi \text{ for some } r_i \downarrow 0.$$

As in the proof of (a), this easily reduces to the corresponding result for $X$ under $\mathbb{P}^*$ and this follows for $p > 1/2$ from (6.15) in Remark 6.7. □

**Remark**. The astute reader will have noticed that whether or not $S(X_1)_2$ is a.s. empty remains unresolved. From Remark 6.7 and the reasoning in (a), we know that the set of points $x$ in $S(X_1)$ satisfying (6.16) has positive $(\log^+ 1/r)^{1-2p+\varepsilon} - m$ for each $\varepsilon > 0$.

## 7. Mass Exponents for Super-Brownian Motion.

Recall the mass exponents $b(q)$ and $B(q)$ defined in (2.7). The underlying measure will always be super-Brownian motion, $X_t$ for $t > 0$. In this section we calculate these exponents. The lower bounds will follow directly from our results on the multifractal spectrum and general theorems from [O] connecting this spectrum to the mass exponents. The upper bounds will require some direct calculation.

By Theorem 4.1

(7.1) $\underline{f}(\alpha) = \dim\{x \in S(X_t) : \underline{d}(X_t, x) = \alpha\} = 2 \cdot 1(\alpha = 2, X_t \neq 0) \quad \mathbb{P}_{m_0}\text{-a.s.} \quad \forall\, t > 0,$

while, by Theorem 6.8, for $t > 0$

(7.2)
$$\overline{f}(\alpha) = \dim\{x \in S(X_t) : \overline{d}(X_t, x) = \alpha\} = \begin{cases} \frac{8}{\alpha} - 2 & \text{if } X_t \neq 0 \text{ and } 2 \leq \alpha \leq 4 \quad \mathbb{P}_{m_0}\text{-a.s.} \\ 0 & \text{otherwise} \end{cases}.$$

If $t > 0$, then $\mathbb{P}_{m_0}$-a.s. on $\{X_t \neq 0\}$

**Theorem 7.2**. (a) $b_{X_t}(q) = 2(1 - q) \quad \forall\, q \in \mathbb{R}$

$$\text{(b) } B_{X_t}(q) = \begin{cases} -4q & \text{if } q \leq -1 \\ 2(1-q) & \text{if } q \geq -1 \end{cases}.$$

**Proof**. As usual [EP] allows us to set $t = 1$ and $m_0 = \delta_0$. $B$ and $b$ are decreasing by Proposition 2.10 of [O] and so an elementary argument shows it suffices to prove the result for each fixed $q \in \mathbb{R}$ (with a null set depending on $q$).

By definition,

$$B(0) = \text{ Dim } \big(S(X_t)\big) = 2 \text{ and } b(0) = \dim\big(S(X_t)\big) = 2 \text{ a.s. on } \{X_t \neq 0\}$$

and so we only need consider $q \neq 0$.

Recall that if $g : \mathbb{R} \to (-\infty, \infty]$, the Legendre transform of $g$ is

$$g^*(\alpha) = \inf_x (\alpha x + g(x)).$$

If $g_1, g_2 : \mathbb{R} \to (-\infty, \infty]$, define

$$g_1 \Box g_2(x) = \begin{cases} g_1(x) & \text{if } x < 0 \\ g_1(0) \vee g_2(0) & \text{if } x = 0 \\ g_2(x) & \text{if } x > 0 \end{cases}.$$

Theorem 2.17(ii) and (iii) of [O] imply

$$\overline{f}(\alpha) \leq (B \Box b)^*(\alpha) \quad 2 < \alpha < 4 \tag{7.3}$$

and

$$\underline{f}(2) \leq (b \Box B)^*(2).$$

These trivially imply

$$\sup_{2<\alpha<4} \overline{f}(\alpha) - \alpha q \leq B \Box b(q) \quad \forall\, q \in \mathbb{R} \tag{7.4}$$

and

$$\underline{f}(2) - 2q \leq b\Box B(q) \quad \forall\, q \in \mathbb{R}. \tag{7.5}$$

(7.5) and (7.1) show that $B(q) \geq 2 - 2q$ for all $q > 0$ a.s. on $\{X_1 \neq 0\}$ while letting $\alpha \downarrow 2$ in (7.4) we see from (7.1) that $B(q) \geq 2-2q$ for all $q < 0$ a.s. on $\{X_1 \neq 0\}$. Finally letting $\alpha \uparrow 4$ in (7.4), we see from (7.2) that $B(q) \geq -4q$ for all $q < 0$ a.s. on $\{X_1 \neq 0\}$. Similarly (7.5) implies $b(q) \geq 2 - 2q$ for $q < 0$ and (7.4) implies $b(q) \geq 2 - 2q$ for $q > 0$ by letting $\alpha \downarrow 2$. This establishes the required lower bounds on $b$ and $B$.

Consider next the upper bounds for $q > 0$. Let $t > 2 - 2q$ and $\varepsilon > 0$. The hitting estimate in Theorem 3.1 of [DIP] shows that if $N_k$ is the number of sets in $\mathcal{D}_k^k$ which intersect $S(X_1)$ then $\mathbb{P}_{\delta_0}(N_k) \leq Ck^d 2^{2k}$ and so by a Borel-Cantelli argument,

$$N_k \leq 2^{(2+\varepsilon/2)k} \text{ for } k \text{ large enough} \quad \mathbb{P}_{\delta_0}\text{-a.s.} \tag{7.6}$$

By Proposition 4.2 there is an $r_0(\omega) > 0$ $\mathbb{P}_{\delta_0}$-a.s. such that

$$\sup_x X_1\big(B(x,r)\big) \leq r^{2-\varepsilon} \quad \text{for } 0 < r < r_0(\omega). \tag{7.7}$$

**Lemma 7.3**. $\mathbb{P}_{\delta_0}$-a.s. for sufficiently large $k$ ($k \geq k_{7.1}(\omega)$ say) if $\{B(x_i, r_i),\ i \in \mathbb{N}\}$ are disjoint sets with $x_i \in S(X_1)$ then the number $M_k$ of these balls such that $r_i > 2^{-k}$ satisfies $M_k \leq 2^{k(2+\varepsilon)}$.

**Proof**. Choose $k$ large so that $S(X_1) \subset [-k,k]^d$ and (7.6) holds. Given a packing as above, for each $B(x_i, r_i)$ with $r_i \geq 2^{-k}$ we may find a $D \subset B(x_i, r_i)$ with $x_i \in D$ and $D \in \mathcal{D}_{k+k_0}^{k+k_0}$ where $k_0 = k_0(d)$ is some geometrical constant. Therefore by (7.6)

$$M_k \leq N_{k+k_0} \leq 2^{(2+\varepsilon)k} \quad \text{for } k \text{ large enough. } \Box$$

Let $\{B(x_i, r_i) : i \in \mathbb{N}\}$ be disjoint balls such that $x_i \in S(X_1)$ and $r_i < 2^{-k_{7.1}(\omega)} \wedge r_0(\omega)$ ($r_0$ as in (7.7)). Then

$$\begin{aligned}
\sum_i X_1\big(B(x_i,r_i)\big)^q (2r_i)^t &\leq \sum_{k > k_{7.1}} \sum_{2^{-k} < r_i \leq 2^{-k+1}} r_i^{q(2-\varepsilon)+t} 2^t && \text{(by (7.7))} \\
&\leq C \sum_{k > k_{7.1}} 2^{(2+\varepsilon)k} 2^{-k\left(q(2-\varepsilon)+t\right)} && \text{(Lemma 7.3)} \\
&\to 0 \text{ as } k_{7.1} \to \infty
\end{aligned}$$

providing $\varepsilon$ is chosen small enough (recall $t > 2 - 2q$). This proves that $\overline{\mathcal{P}}_{X_1}^{q,t}\big(S(X_1)\big) = 0$ and hence (recall (2.8))

$$b_{X_1}(q) \leq B_{X_1}(q) \leq 2 - 2q \quad \text{for } q > 0 \text{ a.s.} \tag{7.8}$$

Now let $q < 0$ and $\delta, \varepsilon > 0$. By Proposition 4.3, for a.a. $\omega$ for each $x$ in $S(X_1)$ there is an $r_x \in (0,\delta)$ such that $X_1\big(B(x,r_x)\big) \geq r_x^{2+\varepsilon}$. By the covering theorem due to Besicovich (1945) there is a covering $\{B(x_i, r_{x_i}) : i \in \mathbb{N}\}$ of $S(X_1)$ which may be partitioned into $K(d)$ sets of disjoint balls. If $\{B(x_i, r_{x_i}) : i \in I_j\}$ $j = 1, \ldots, K(d)$ are these disjoint sets then for $t > 2 - 2q$

$$\begin{aligned}
\sum_i X_1\big(B(x_i, r_{x_i})\big)^q (2r_{x_i})^t &\leq \sum_{j=1}^{K(d)} \sum_{k \geq \log(1/\delta)} \sum_{i \in I_j, 2^{-k} < r_{x_i} \leq 2^{-k+1}} (r_{x_i})^{q(2+\varepsilon)+t} 2^t \\
&\leq CK(d) \sum_{k \geq \log(1/\delta)} 2^{k[(2+\varepsilon)-q(2+\varepsilon)-t]} \quad \text{(Lemma 7.3)} \\
&\to 0 \quad \text{as } \delta \downarrow 0
\end{aligned}$$

providing $\varepsilon$ is chosen small enough. This proves that $\overline{\mathcal{H}}^{q,t}_{X_1}\big(S(X_1)\big) = 0$ and, as the same argument applies to subsets of $S(X_1)$, this proves $\mathcal{H}^{q,t}_{X_1}\big(S(X_1)\big) = 0$ and hence $b_{X_1}(q) \leq 2 - 2q$ for $q < 0$.

It remains only to establish the upper bound on $B_{X_1}(q)$ for a fixed $q < 0$. It suffices to consider packings using semi-dyadic cubes as in Corollary 2.2, and since $S(X_1)$ is a.s. compact we may fix $K_0 \in \mathbb{N}$ and consider semi-dyadic cubes in $\mathcal{D}_n^{K_0}$ for some $n$. Let $\varepsilon > 0$. Then $\mathbb{P}_{\delta_0}$-a.s. for $n$ sufficiently large

$$\begin{aligned}
\sum\nolimits_n &\equiv \sum_{D \in \mathcal{D}_n^{K_0}} 1\big(X_1(D^*) > 0\big) X_1(D)^q 2^{-nt} \\
&\leq \sum_{k=1}^{\infty} \sum_{D \in \mathcal{D}_n^{K_0}} 1\left(0 < X_1(D^*), 2^{-k-1} \leq X_1(D) < 2^{-k}\right) 2^{-kq-nt} 2^{-q} \\
&\leq c 2^{-2n(1+\varepsilon)q-nt} N_n + c \sum_{k \geq 2n(1+\varepsilon)} L_n^{K_0}(k/n) 2^{-kq-nt},
\end{aligned}$$

where we recall the notation used in the proof of proposition 5.1 for $\beta$-light semi-dyadic cubes. (5.1) shows that $L_n^{K_0}\big(4(1+\varepsilon)\big) = 0$ for large $n$ a.s. This and (7.6) imply that a.s. for large $n$

$$\begin{aligned}
\sum\nolimits_n &\leq c 2^{-2n(1+\varepsilon)(q-1)-nt} + c \sum_{2n(1+\varepsilon) \leq k \leq 4n(1+\varepsilon)} L_n^{K_0}(k/n) 2^{-kq-nt} \\
&\equiv c 2^{-2n(1+\varepsilon)(q-1)-nt} + \sum\nolimits_{2,n}.
\end{aligned} \tag{7.9}$$

(5.1) shows that

$$\begin{aligned}
\mathbb{P}_{\delta_0}\left(\sum\nolimits_{2,n}\right) &\leq c(K_0) \sum_k 1\big(2(1+\varepsilon) \leq k/n \leq 4(1+\varepsilon)\big) 2^{-n(k/n-4)-kq-nt} \\
&\leq \begin{cases} c(K_0) 2^{n(4-t)-2n(1+q)(1+\varepsilon)} & \text{if } q > -1 \\ c(K_0) 2^{n(4-t)-4n(1+q)(1+\varepsilon)} & \text{if } q \leq -1 \end{cases}.
\end{aligned}$$

If $q > -1$ and $t > 2(1-q)$ or $q \leq -1$ and $t > -4q$ we may choose $\varepsilon$ small enough so that the above bound is summable over $n$ and so

$$\sum_n \sum\nolimits_{2,n} < \infty \text{ a.s.}$$

If $t > 2(1-q)$, we may again may choose $\varepsilon > 0$ small enough so that the first term in (7.9) is also summable over $n$. Apply Corollary 2.2 to conclude that

$$B_{X_1}(q) \leq \begin{cases} 2(1-q) & \text{for } q > -1 \\ -4q & \text{for } q \leq -1 \end{cases} \quad \text{a.s. } \square$$

Finally, let us briefly discuss the packing dimension of

$$C_\alpha(X_1) = \{x \in S(X_1) : \overline{d}(X_1, x) \geq \alpha\}.$$

Since $S(X_1)_{\alpha/2} \subset C_\alpha(X_1)$ for $\alpha > 2$ (Corollary 4.4) and $S(X_1)_{\alpha/2}$ is $\alpha/2$-thin, Lemma 2.6 implies

$$\begin{aligned} \text{Dim } C_\alpha(X_1) \geq \text{ Dim } \big(S(X_1)_{\alpha/2}\big) &\geq \frac{\alpha}{2} \dim\big(S(X_1)_{\alpha/2}\big) \\ &= 4 - \alpha \qquad \alpha \in [2,4] \text{ a.s. (Theorem 6.8).} \end{aligned}$$

(Note if $\alpha = 2$ the above inequality is immediate from Theorem 6.8.)

**Conjecture 7.4**. Dim $C_\alpha(X_1) = 4 - \alpha$ for $\alpha \in [2,4]$ a.s.

As

$$B^*_{X_1}(\alpha) = \begin{cases} 4-\alpha & 2 \leq \alpha \leq 4, \\ -\infty & \text{otherwise} \end{cases}$$

if (7.3) could be extended to

$$\text{Dim } C_\alpha(\mu) \leq B^*_\mu(\alpha),$$

then the conjecture would follow. Such a result would evidently be rather useful as a direct verification of Conjecture 7.4 appears to be difficult.

## REFERENCES


[BEP] Barlow, M.T., Evans, S.N., and Perkins, E.A. (1991) Collision Local Times and Measure-valued Processes, *Can. J. Math.* 43, 897–938.

[B] Besicovich, A.S. (1945) A general form of the covering principle and relative differentiation of additive functions, *Proc. Camb. Phil. Soc.* 41, 103-110.

[C] Cutler, C.D. (1992) Measure disintegrations with respect to a $\sigma$-stable monotone index and the pointwise representation of packing dimension, *Rend del Circolo Math di Palermo* 28, 319–340.

[D1] Dawson, D.A. (1992) Infinitely divisible random measure and superprocesses, in *Proc. 1990 Workshop on Stochastic Analysis and Related Topics*, Silivri, Turkey.

[D2] Dawson, D.A. (1993) Measure-valued Markov Processes, *Ecole d'Eté de Probabilités de Saint-Flour XXI*, Lecture Notes in Math. 1541 1–260. Springer, Berlin.

[DIP] Dawson, D.A., Iscoe, I., and Perkins, E.A. (1989) Super-Brownian motion: Path properties and hitting probabilities, *Probab. Theory Related Fields* 83, 135–206.

[DP] Dawson, D.A. and Perkins, E.A. (1991) *Historical Processes*, Memoirs of the Amer. Math. Soc. 93 no. 454, 179 p.

[EP] Evans, S.N. and Perkins, E.A. (1991) Absolute continuity results for superprocesses with some applications, *Trans. Amer. Math. Soc.* 325, 661–681.

[F] Falconer, K.J. (1990) *Fractal Geometry: mathematics foundations and applications*, Wiley, New York.

[Fr] Frostman, O. (1935) Potentiel d'équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions, *Medd Lunds Univ. Math. Sem.* 3, 1–188.

[Ha] Halsey, T.C., Jensen, M.H., Kadanoff, L.P., Procaccia, I., and Shraiman, B.I. (1986) Fractal measures and their singularities, *Phys. Rev.* A33, 1141–1151.

[K] Knight, F.B. (1981) Essentials of Brownian Motion and Diffusion, *American Math. Soc.*, Providence.

[LG1] Le Gall, J.F. (1993) A class of path-valued Markov processes and its applications to superprocesses, *Probab. Theory Related Fields* 95, 25–46.

[LG2] Le Gall, J.F. (1994a) A path-valued Markov process and its connection with partial differential equations, *Proceedings of the First European Congress of Mathematics*, 185–212, Birkhäuser, Boston.

[LG3] Le Gall, J.F. (1994b) A lemma on super-Brownian motion with some applications, *Festschift in Honor of E.B. Dynkin* (M. Friedlin, ed.), 237–251, Birkhäuser, Boston.

[LPT] Le Gall, J.F., Perkins, E.A., and Taylor, S.J. (1995) The packing measure of the support of super-Brownian motion, *Stoch. Proc. Appl.* 59, 1–20.

[O] Olsen, L. (1995) A multifractal formalism, *Advances in Mathematics* 116, 82-195.

[P1] Perkins, E.A. (1989) The Hausdorff measure of the closed support of super-Brownian motion, *Ann. Inst. H. Poincaré* 25, 205–224.

[P2] Perkins, E.A. (1992) Measure-valued branching diffusions with spatial interactions, *Probab. Theory Related Fields* 94, 189–245.

[PT] Perkins, E.A. and Taylor, S.J. (1987) Uniform measure results for the image of subsets under Brownian motion, *Probab. Theory Related Fields* 76, 257–289.

[T] Taylor, S.J. (1961) On the connection between generalized capacities and Hausdorff measures, *Proc. Cambridge Philos. Soc.* 57, 524–531.

[TT] Taylor, S.J. and Tricot, C. (1985) Packing measure and its evaluation for a Brownian path, *Trans. Amer. Math. Soc.* 288, 679–699.

[Tr] Tribe, R. (1991) The connected components of the closed support of super-Brownian motion, *Probab. Theory Related Fields* 89, 75–87.

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