Wiener Measure

\[ W^d = C([0, \omega_1], 1^{\omega_1}) \times \{ \omega : \omega \in \Omega \} \to 1^{\omega_1} \text{ of } \omega \text{-continuous } \sigma \]

Set \( Y^d = \omega \bmod \delta \) and \( W^d = \delta (Y^d : b(30)) \) a \( \sigma \)-field on \( W^d \).

If \( w, w_1, w_2 \in W^d \), let \( d(w, w_2) = \frac{1}{\sqrt{n}} \sup_{b \in \mathbb{N}} \left| \frac{1}{n} \sum_{b=1}^{n} Y^d(b) - \frac{1}{n} \right| \leq \frac{2}{n} \sum_{b=1}^{n} \left| w(b) - w_2(b) \right| \).

Then \( (W^d, d) \) is a metric space, and \( W_n \overset{d}{\to} w \iff \{ u_n \to u \} \) uniformly on compact sets.

Prop. 5.1: \( W^d = \theta(W^d) = \theta \left( \bigcup U : U \subseteq W^d, U \text{ open } \right) \).

(1) \( y \to Y^d(y) \) is continuous, hence \( \theta(W^d) \)-measurable.

\[ W^d = \theta(Y^d : b(30)) \subseteq \theta(W^d) . \]

For the converse inclusion, fix \( w \in W^d, n \in \mathbb{N} \).

\[ d_n(w, w) = \sup_{b \in \{1, \ldots, n \}} \left| \frac{1}{n} \sum_{b=1}^{n} Y^d(b) - \frac{1}{n} \right| \text{ is } W^d \text{-measurable in } W . \]

By definition, \( W \to d(w, w) \) is \( W^d \)-measurable.

\[ \theta(W^d) \subseteq \theta \left( \bigcup_{b \in \mathbb{N}} \{ w : d_n(w, w) \leq \varepsilon \} \text{ a.e. } w \in W^d \right) . \]

\( (W^d, d) \) is separable ("polynomials" with rational coefficients are dense).

Every open set, being a countable union of open balls, is in \( W^d \).

\[ \theta(W^d) \subseteq \theta(W^d) . \]

\( W^d_0 = \text{Field of finite dimensional sets in } W^d \)

\[ \{ A \in W^d_0 : \text{a.e. } \omega \} \to = \omega \subseteq \{1, \ldots, n\}, \quad C \in \theta \left( \mathbb{R}^{(\omega)} \right) \} \}

\[ \theta(W^d_0) = \theta(W^d) . \]

Let \( B^d \in \mathbb{N} \) be a \( d \)-dimensional Brownian motion on \( (\Omega, \mathcal{F}) \) with initial law \( \mu \). Redefine \( B^d \) be \( \mathcal{G} \) on a well set \( \omega \) so that \( B^d(\omega) \) is cont's \( \sigma \).

Define \( B : \mathbb{R} \to W^d \). Then \( B^d(A) \subseteq \mathcal{G} \quad \forall \ A \in W^d \)

\[ w \to B^d(w) \Rightarrow B \text{ is measurable map from } \mathbb{R} \text{ to } W^d \]

Define \( p^d \) on \( (W^d, W^d) \) by \( p^d(A) = P(B^d(A)) \).

For \( A \in W^d_0 \) a.e. \( \omega \), \( p^d(A) = P \left( \omega : (B_{\omega(1)}, \ldots, B_{\omega(m)}) \in \mathbb{C} \right) \)

\( \text{(by PPD)} \)

\[ = \sum_{\mathbb{C} \in \mathbb{C} \{m \ldots m \}} \prod_{i=1}^{m} \left( p^d_{B_{\omega(i)}, B_{\omega(i+1)}} (x_i - x_{i+1}) \right) dx_1 \ldots dx_m dp_{\omega(1), \ldots, \omega(m)} . \]
This shows that $p_t^W(X)$ does not depend on the choice of $B$. (New $W^B_t$)

By Bessel's formula, $p_t^W$ uniquely determines $p_t$.

$p_t^W$ does not depend on the choice of $B$.

**Def.** We call $p_t^W$ Wiener measure with initial law $\mu$. 

So $p_t^W = p_t^\mu$, $p_0^\mu$ is called Wiener measure.

**Remark.** On $(W_0^W, \mathcal{F}^W, p_t^W)$, if $W_t^W = \mathcal{F}^W_t = \{ \sigma_{s \leq t} \}$, then $p_t^W = \mathcal{F}^W_t$.

Then $p_t^W = \mathcal{F}^W_t$ is a $W_t^W$-Brownian motion because it has the same law as $B$ which is on $\mathcal{F}^W_t$ - Brownian motion.

We call $(W_t^W, \mathcal{F}^W_t, p_t^W, \mathcal{F}^W_t, \mathcal{P}^W_t)$ the canonical representation of Brownian motion.

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**Brownian Semigroup**

Let $p_t = \frac{1}{2 \pi t} e^{-x^2/(2t)}$, in our old notation.

The semigroup property now becomes:

$$p_t(x, y) = \int p_s(x, z) p_t(z, y) dz$$

For $f \in \mathcal{B}(R^d)$, $t > 0$, let $P_t f(x) = (\int f(y) p_t(x, y) dy)$. Then $x \to P_t f(x)$ is in $\mathcal{B}(R^d)$. (For measurability note that for $f \in L^2(\mathbb{R}^d)$ and use Monotone Class Thm.)

$$P_t : \mathcal{B}(R^d) \rightarrow \mathcal{B}(R^d)$$

$$p_t f(x) = \int f(y) p_t(x, y) dy = \int \left( p_t(x, z) \int f(y) p_t(z, y) dy \right) dz = \int p_t(x, z) f(z) dz = p_t f(z)$$

$$p_{s+t} f(x) = p_s p_t f(x)$$

Call $(P_t)_{t \geq 0}$ the Brownian semigroup.
Thm 5.17. (Markov Property) Let $B$ be a d-dimensional Brownian motion.

Let $g, h : \mathbb{R} \to \mathbb{R}$ be bounded measurable functions.

\[
P_r \{ g(B_t) \, | \, \mathcal{F}_s \} \left( B_t \in A \right) = P_r \left\{ g(B_t) \in A \right\}.
\]

P1) Recall: $Y, Z$ independent of $\mathcal{F}_s$, $\alpha$ bounded measurable.

\[
E \left[ \alpha(Y, Z) 1_{B_s \in A} \right] = \int \alpha(y, z) P(A \mid B_s = y) \, dP(y).
\]

So

\[
\text{LHS} = E \left( \frac{E(B_{t+\delta}. - B_t)}{\sqrt{\delta}} \right) \mid \mathcal{F}_s = \int \frac{E(B_{t+\delta}. - B_t)}{\sqrt{\delta}} \, dP(y).
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(ML, iid)

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