1. (a) Let \( f : \mathbb{R}_+ \to \mathbb{R} \) be continuous and \( B \) be a standard 1-dimensional \((\mathcal{F}_t)\)-Brownian motion. Show that \( I_t = \int_0^t f(s) dB_s \) is a mean 0 normal r.v. with variance \( \int_0^t f(s)^2 ds \).

**Hint.** One approach is to use Ito's Lemma to find the characteristic function of \( I_t \).

(b) Consider the SDE: \( X_t = x + \sigma B_t - \lambda \int_0^t X_s ds \), where \( \sigma, \lambda > 0 \), \( x \in \mathbb{R} \) and \( B \) is as above. Show \( X \) has a unique solution and show that it is given by \( X_t = \sigma e^{-\lambda t} \int_0^t e^{\lambda s} dB_s + xe^{-\lambda t} \).

(c) Show that \( X_t \) converges in distribution as \( t \to \infty \) and find the limiting distribution.

2. **Not to hand in.**

(a) Let \( X \) be a non-negative supermartingale and \( S \leq T \) be uniformly bounded stopping times. Show that \( E(X_T | \mathcal{F}_S) \leq X_S \) a.s. This result holds without the non-negativity hypothesis but this simplifies the proof a bit (feel free to prove this without the non-negativity assumption).

(b) Let \( X \) be as above and \( T \) be the first time \( t \) that \( X_t = 0 \) (\( T \leq \infty \)). Prove that \( X_t = 0 \) for all \( t \geq T \) a.s.

3. Let \( B \) be a standard one-dimensional \((\mathcal{F}_t)\)-Brownian motion, \( a, b > 0 \), let \( P \) denote its law, and let \( \tau_a = \inf\{t \geq 0 : B_t = a\} \).

(a) Let \( Q_b \) be the law of \( X_t = B_t + bt \) on \((W, \mathcal{W}) = (W^1, \mathcal{W}^1)\). Prove that \( P \) and \( Q_b \) are equivalent laws on each \( \mathcal{F}_t^{0,X} \) but are not equivalent on \( \mathcal{F}_\infty^{0,X} \).

(b) The reflection principle implies that

\[
P(\tau_a \leq t, B_t \in dy) = \left[ \int_0^t (a/s)p_s(a)p_{t-s}(y-a)ds \right]dy.
\]

Convince yourself (but not me) that this is “obvious”. Use this and the Cameron-Martin-Girsanov formula to show that the density function of \( \tau_{a,b} = \inf\{t \geq 0 : B_t = a - bt\} \) is \((a/t)p_t(a-bt)\). [Here \( p_t(x) = (2\pi t)^{-1/2}e^{-x^2/2t} \].

4. If \( M \in c\mathcal{M}_{0,1} \) show that for any \( t, x, y > 0 \),

\[
P(M_t^* \geq x, [M]_t \leq y) \leq 2 \exp(-x^2/2y).
\]

Conclude that if \([M]_t \leq ct\) for all \( t \), then for all \( a, t > 0 \), \( P(M_t^* \geq at) \leq 2 \exp(-a^2t/2c) \).

**Hint.** Recall our proof of Lemma 2.7–the exponential bounds for Brownian motion.