Math 546 Assignment 6 (due noon Dec. 12
in my mailbox, under my door or sent to perkins@math.ubc.ca)

1. This question is a revisit to the Helms-Johnson example but the process is now a bit different, and recall we did not prove a couple of the key lemmas (which will be addressed below). Your arguments below should be self-contained–feel free to use any of the arguments we did earlier but do not quote the results.

Let $B$ be a 3-dimensional $(\mathcal{F}_t)$-Brownian motion starting at $x \neq 0$ under the probability $P_x$. Expectation with respect to $P_x$ is denoted by $E_x$. Let $T_a = \inf\{t \geq 0 : |B_t| \leq a\}$, $S_b = \inf\{t \geq 0 : |B_t| \geq b\}$ and $X(t) = |B(t)|^{-1}$.

(a) *Don’t hand in this part.* If $f : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ is a $C^2$ function and $Y$ is a 3-dimensional continuous semimartingale s.t. $Y_t \neq 0$ for all $t$, convince yourself, but not me, that we may apply Ito’s formula to $f(Y_t)$. [It suffices to consider $Y_{T_\varepsilon}$ where $T_\varepsilon$ is the hitting time of $\{|x| \leq \varepsilon\}$ by $Y$, and let $\varepsilon \downarrow 0$. For the stopped semimartingale note that one can find a $C^2$ function, $g$, on the entire plane which agrees with $f$ on the complement of $B(0, \varepsilon/2)$.

(b) Show for $0 < p < 3$ there is a $c = c_p$ so that $E_x(|B_t|^{-p}) \leq c \min(|x|^{-p}, t^{-p/2})$ for all $x \in \mathbb{R}^3 \setminus \{0\}$, $t > 0$. (The argument below also works for $x = 0$ but we will be assuming $x$ is non-zero.)

**Hint:** One approach is direct calculation of $\int |z|^{-p} \exp(-|z - x|^2/2t)(2\pi t)^{-3/2}dz$. E.g., for the $c|x|^{-p/2}$ upper bound, one can split the integral into $|z| < |x|/2$ and $|z| \geq |x|/2$, and for the $ct^{-p/2}$ upper bound one can use scaling to reduce it to $t = 1$.

(c) For $a < |x| < b$, show that $X(t \land T_a \land S_b)$ is an $(\mathcal{F}_t)$-martingale and use it to find $P_x(T_a < S_b)$. Now show that $P_x(B_t = 0 \text{ for some } t \geq 0) = 0$.

**Hint:** Ito’s Formula.

(d) Show $X_t$ is an $L^2$-bounded (hence uniformly integrable) continuous $(\mathcal{F}_t)$-local martingale and $(\mathcal{F}_t)$-supermartingale under $P_x$.

(e) Prove that $\lim_{t \to \infty} |B_t| = \infty$ $P_x$-a.s. Show that $X$ is not an $(\mathcal{F}_t)$-martingale.

2. Let $B$ be a standard one-dimensional $(\mathcal{F}_t)$-Brownian motion. Prove there is no solution to the SDE

$$X_t = \int_0^t 1(X_s \geq 0)dB_s$$

on any set-up.

**Hint.** If $\phi(x) = -x^31(x < 0)$, consider $\phi(X_t)$.

3. Let $X$ be a non-negative $(\mathcal{F}_t)$-martingale.

(a) Let $S \leq T$ be uniformly bounded $(\mathcal{F}_t)$-stopping times. Show that $E(X_T|\mathcal{F}_S) = X_S$ a.s.

(b) Let $S = \inf\{t \geq 0 : X_t = 0\}$, where $\inf \emptyset = \infty$. Prove that $X_t = 0$ for all $t \geq S$ a.s.

**Hint:** For $\varepsilon > 0$, consider $T = \inf\{t > S : X_t \geq \varepsilon\}$. Of course $S$ and $T$ need not be bounded!

This result holds for non-negative super-martingales, with the obvious change in (a).