1. Let $Z$ be a continuous non-negative $(\mathcal{F}_t)$-local martingale such that $E(Z_0) < \infty$. Show that:
   (a) $Z$ is an $(\mathcal{F}_t)$-supermartingale and $Z_t \to Z_\infty$ a.s.
   (b) $Z$ is an $(\mathcal{F}_t)$-martingale iff $E(Z_t) = E(Z_0)$ for all $t > 0$.
   (c) $Z$ is a uniformly integrable $(\mathcal{F}_t)$-martingale iff $E(Z_\infty) = E(Z_0)$.

2. (a) Let $f : \mathbb{R}_+ \to \mathbb{R}$ be Borel measurable and satisfy $\int_0^t f(s)^2 \, ds < \infty$ for all $t > 0$, and $B$ be a standard 1-dimensional $(\mathcal{F}_t)$-Brownian motion. Show that $Z_t = \int_0^t f(s)dB_s$ is a mean 0 normal r.v. and find its variance.
   **Hint.** Consider $Y(t) = \exp[i\theta Z_t + (\theta^2/2) \int_0^t f(s)^2 \, ds]$ and use Ito's Lemma.

   (b) Consider the Stochastic Differential Equation: $X_t = x + \sigma B_t - \lambda \int_0^t X_s \, ds$, where $\sigma, \lambda > 0$, $x \in \mathbb{R}$ and $B$ is as above. Show $X_t$ has a unique solution and show that it is given by $X_t = \sigma e^{-\lambda t} \int_0^t e^{\lambda s} dB_s + x e^{-\lambda t}$. Do this by considering $e^{\lambda t} X_t$.

   (c) Show that $X_t$ converges in distribution as $t \to \infty$ and find the limiting distribution.

3. (Tanaka’s formula and local time). Let $B$ be a standard 1-dimensional $(\mathcal{F}_t)$-Brownian motion. For every $\varepsilon > 0$ set $g_\varepsilon(x) = \sqrt{\varepsilon + x^2}$ for $x \in \mathbb{R}$. Below you will show that Ito’s lemma remains valid in an appropriate sense for Brownian motion and the non-$C^2$ function $f(x) = |x|$.

   (a) Show that $g_\varepsilon(B_t) = \sqrt{\varepsilon} + M^\varepsilon + A^\varepsilon$, where $M^\varepsilon$ is a square integrable martingale and $A^\varepsilon$ is a continuous increasing process. Give explicit formulae for $M^\varepsilon$ and $A^\varepsilon$.

   (b) Set $sgn(x) = 1_{\{x > 0\}} - 1_{\{x < 0\}}$. Show that for for every $T > 0$

   $$\sup_{t \leq T} |M^\varepsilon_t - \int_0^t sgn(B_s)dB_s| \to 0 \text{ in } L^2 \text{ as } \varepsilon \to 0.$$  

Conclude that there is a continuous increasing process $L$ such that

$$|B_t| = \int_0^t sgn(B_s)dB_s + L_t \text{ for all } t \geq 0,$$

and for all $T > 0$,

$$\sup_{t \leq T} |A^\varepsilon(t) - L(t)| \to 0 \text{ in } L^2 \text{ as } \varepsilon \to 0.$$  

(c) Show that $\beta_t = \int_0^t sgn(B_s)dB_s$ is an $(\mathcal{F}_t)$-Brownian motion.

(d) Prove that with probability 1, for all $0 \leq u < v$, $B \neq 0$ on $(u, v)$ implies $L_v = L_u$. Hence $t \to L_t$ only increases on the zero set of $B$.

   **Hint:** Explain why it suffices to show that for fixed rationals $p < q$, w.p. 1 $B \neq 0$ on $[p, q]$ implies $L_p = L_q$.

(e) Show that $L_t = \sup_{s \leq t} -\beta_s$ for all $t$ a.s., and conclude that $L_t > 0$ for all $t > 0$ a.s.

   **Hint:** In order to show $L_t \leq \sup_{s \leq t} -\beta_s$ consider $L_{\alpha_t}$, where $\alpha_t = \sup\{s \leq t : B_s = 0\}$, and recall (d). The reverse inequality is easy to prove.