Math 546 Assignment 5 (due April 2)

1. Verify that the Brownian semigroup \( P_t f(x) = \int f(y)p_t(y - x)dy \) is a Feller semigroup. Here
\[
p_t(z) = \exp\{-\|z\|^2/2t\}(2\pi t)^{-d/2}
\]
is the \( d \)-dimensional density of Brownian motion. (The fact that \( p_t * p_s = p_{s+t} \) gives the semigroup property. You need to verify the other properties.)

2. Prove Proposition 5.12: Let \( Z \) be a continuous non-negative \((\mathcal{F}_t)\)-local martingale such that \( E(Z_0) < \infty \). Show that:
   (a) \( Z \) is an \((\mathcal{F}_t)\)-supermartingale and \( Z_t \to Z_\infty \) a.s.
   (b) \( Z \) is an \((\mathcal{F}_t)\)-martingale iff \( E(Z_t) = E(Z_0) \) for all \( t > 0 \).
   (c) \( Z \) is a uniformly integrable \((\mathcal{F}_t)\)-martingale iff \( E(Z_\infty) = E(Z_0) \).

3. Let \( U \) be an open subset of \( \mathbb{R}^d \) and \( X \) be a continuous semimartingale taking values in \( U \) for all \( t \geq 0 \) a.s. If \( f \in C^2(U) \), then Itô’s lemma continues to apply to \( f(X_t) \). Convince yourself, but not me, that our proof can be adapted to this setting by considering the process stopped when it gets within a distance \( 1/n \) of \( U^c \).

   Let \( B \) be a 3-dimensional Brownian motion starting at a non-zero point \( x \). Let \( T_a = \inf\{t \geq 0 : |B_t| \leq a\} \) and \( S_b = \inf\{t \geq 0 : |B_t| \geq b\} \). Let \( X(t) = |B(t)|^{-1} \).

   (a) For \( a < |x| < b \), show that \( X(t \wedge T_a \wedge S_b) \) is a martingale and use it to find \( P(T_a < S_b) \).

   Now show that \( P(B_t = 0 \text{ for some } t \geq 0) = 0 \).

   (b) Show \( X_t \) is an \( L^2 \)-bounded (hence uniformly integrable) local martingale and supermartingale, but is not a martingale.

   (c) Prove that \( \lim_{t \to \infty} |B_t| = \infty \) a.s.