Math 546 Assignment 3 (due Feb. 28)

1. Assume \( X \) is an \((\mathcal{F}_t)\)-Poisson process with rate \( \lambda > 0 \) and let \( M_t = X_t - \lambda t \). Prove that \( M \) and \( M^2 - X \) are \((\mathcal{F}_t)\)-martingales. Recall this completes the proof that for any constant \( K > 0 \), \([M^K] = X^K\).

2. Show there is a constant \( c_4 \) so that for any martingale \( \{M_n: n \in \mathbb{Z}_+\} \), if \( d_n = M_n - M_{n-1} \) and \([M] = \sum_{n=1}^{\infty} d_n^2\) then \( E([M]^{2n}) \leq c_4 E((M^*)^4) \).

Hint. \( (\sum_{n=1}^{\infty} d_n^2)^2 = 2 \sum_{j=1}^{\infty} d_j^2 \sum_{i=j+1}^{\infty} d_i^2 + \sum_{j=1}^{\infty} d_j^4 \).

3. Let \( M \in \mathcal{M}_{0,loc} \). If \( E(M^*) < \infty \), show that \( M \) is a u.i. \((\mathcal{F}_t)\)-martingale

Here are a couple of easy questions.

4. If \( M, N \in \mathcal{M}_{0,loc}^2 \) or \( M, N \in c\mathcal{M}_{0,loc} \), define \([M, N] = 1/4([M + N] - [M - N])\) and let
\[
A_n(t) = \sum_{i=1}^{2^n} (M(t^n_i) - M(t^n_{i-1}))(N(t^n_i) - N(t^n_{i-1})),
\]
where \( t^n_i = i 2^{-n} \wedge t \).

(a) [Don’t hand in.] If \( M, N \in \mathcal{M}_{0,loc}^2 \), show that \([M, N]\) is the unique process in \( \mathcal{F}_V \) s.t.

(i) \( MN - [M, N] \) is a u.i. \((\mathcal{F}_t)\)-martingale, and

(ii) \( \Delta [M, N]_t = (\Delta M)_t (\Delta N)_t \).

Show also that \( A_n(t) \to [M, N]_t \) in probability.

(b) If \( M, N \in c\mathcal{M}_{0,loc} \), show that \([M, N]\) is the unique process in \( \mathcal{F}_V \) s.t. \( MN - [M, N] \in c\mathcal{M}_{0,loc} \). Show also that for any \( K > 0 \), \( \sup_{t \leq K} |A_n(t) - [M, N]_t| \to 0 \) in probability.

5. Read Sec. 28 (p. 50-51) on the Kunita-Watanabe inequality. Use (28.3) on p. 50 to prove that

**Theorem 4.23.** If \( M, N \in \mathcal{M}_{0,loc}^2 \), \( H \in L^2(M) \) and \( K \in L^2(N) \), then

\[
[H \cdot M, K \cdot N]_t = \int_0^t H_s K_s d[M, N]_s.
\]