Math 54b Solutions to HW 1

1. Fix outside a null set \( x_\epsilon = y_\epsilon \cap \partial C \).
   By right continuity \( x_\epsilon = y_\epsilon \cap \partial C \), i.e. \( x_\epsilon \) and \( y_\epsilon \) are
   indistinguishable.

2. We only need show \( x_\epsilon \subset y_\epsilon \) as this gives \( y_\epsilon \subset x_\epsilon \) and \( \mathbb{N}^1 \) is clear.
   Let \( b_n, b \) and \( A \in x_\epsilon = \bigcap_{n} y_\epsilon \). Let \( s_n = e_n + \frac{1}{n} \). Let \( k \).

   By the given hint \( \exists b_n \in x_\epsilon \subset y_\epsilon \) and \( N_1, N_2 \in \mathbb{N}^1 = \mathbb{N}^2 \).
   \[
   A = (B_n \cup N_2^1) - N_2^2
   \]
   Let \( B = \lim_n b_n \). Then \( B_n \in \bigcap_{n} y_\epsilon \).

3. \( B - A = \bigcap_{n} (B_n - A) = \bigcap_{n} N_2 = N_2 \) but \( N_2 \cap \mathbb{N}^1 = \mathbb{N}^2 \). \( \mathbb{N}^2 \).

   \[
   A = (B \cup N_2^1) - N_2^2 \text{ where } N_1 = A - B \in \mathbb{N}^1, N_2 = B - A \in \mathbb{N}^2,
   \]

3. For \( \epsilon < \delta \) and \( A \in \mathbb{N}^2 \),
   \[
   E(\epsilon^{10}(B_\epsilon - B_\delta)|A) = \lim_{n \to \infty} E(\epsilon^{10}(B_\epsilon - B_\delta) | \mathbb{N}_n^1) / P(A')
   \]

(3)

\[
= \lim_{n \to \infty} E(\epsilon^{10}(B_\epsilon - B_\delta) | \mathbb{N}_n^1) / P(A')
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= \lim_{n \to \infty} E(\epsilon^{10}(B_\epsilon - B_\delta) | \mathbb{N}_n^1) / P(A')
\]

\[
= e^{-\epsilon^{10}(B_\epsilon - B_\delta)} = E(e^{10(B_\epsilon - B_\delta)}) \text{ (Uniqueness)}
\]

(3) \( P(B_\epsilon - B_\delta \in C | A) = P(B_\epsilon - B_\delta \in C) \forall \epsilon \in \mathbb{N}^1 \) (Prop. of \( C^2 \))

(3) \( P(B_\epsilon - B_\delta \in C | A) = P(B_\epsilon - B_\delta \in C) P(A) \). This is trivial if \( P(A) = 0 \)

So (3) holds \( \forall A \in \mathbb{N}^1 \).
- $B_6 - B_5$ is independent of $\chi_b$ and has a $N(0, (6-4) I_6)$ law.

$B_6$ is $\chi_b$-mile since $\chi_b \leq \chi_b$

- $B$ is an $\mathcal{M} - \mathcal{B}$ (continuity is immediate).

4. Let $s < t$.

$$E(M_t - M_s | \mathcal{F}_s) = E((B_t - B_s)^3 + 2(B_t - B_s) B_s I_7^b) + (-6-3)$$

$$= E((B_t - B_s)^3 | \mathcal{F}_s) + 2 B_s E(B_t - B_s) - (6-3)$$

$- t - s + 0 - (t - s) = 0.$

$M$ is only $\chi_b^4$-martingale (adaptedness and
continuity is obvious).

5. Continuity of $t \to M_t$ is immediate.

5. Recall from Cor 3.12 that

$$P(M_t > R) \leq e^{-R^2/2t}.$$

$$P(M_{n+1} > \text{const} n) \leq \exp \left(-\frac{c^2 \text{const} n^2}{2 n^2}\right)$$

$$\sum_{n=1}^{\infty} P(M_{n+1} > \text{const} n) < \infty$$

By Bonnet-Cardell, $\exists N(k) < c_0$ s.t. $M_{n+1} = \text{const} n)$ $\forall n \geq N$.

Let $0 < t = r^{n-1}$, choose $n > k$ s.t. $r^{n-1} < r^n$,

$$B_t \leq M_{n+1} = \text{const} n) \leq c_0 t$$

$$\lim_{n \to \infty} B_t / \text{const} n) = c_0 \text{ a.s.} \text{ (c > 1).}$$

Take $c = c_0 t$ to conclude $\lim_{n \to \infty} B_t / \text{const} n) = c \text{ a.s.}$.
6. By the reverse submartingale property, for $k \leq n$,

$$E[X_n I(X_n \leq -\varepsilon)] = E[X_n] - E[X_n I(X_n > -\varepsilon)]$$

$$= E[X_n] - E[X_n] + E[X_k I(X_k \leq -\varepsilon)]$$

$$= E[X_n] - E[X_k] + E[X_k I(X_k \leq -\varepsilon)]$$

So, for $k \leq n$,

$$E[I(X_n) I(X_n \geq 0)] = E[X_n I(X_n \geq 0)] - E[X_n I(X_n \leq -\varepsilon)]$$

$$\leq E[X_n I(X_n \geq 0)] - (E[X_n] - E[X_k]) \quad (\text{by } \delta)$$

$$+ E[X_k I(X_k \leq -\varepsilon)]$$

$x_n^+ \rightarrow x_0^+$ is a submartingale, so that

$$E[X_n^+] \leq E[X_0^+] \quad \text{all } n \geq 0.$$

By hypothesis, $\exists C > 0$ s.t. $E[X_n^+] - E[X_k^+] \geq -C \forall n$ and so by

$$E[I(X_n)] = E[X_n^+] + E[X_k^+] \leq 2E[X_n^+] + C \leq 2E[X_0^+] + C$$

By Rev. Subm. Prop. and $E[X_n]$ is a finite. Let $\varepsilon > 0$, choose $k_0$ s.t. $n \geq k_0 \Rightarrow |E[I(X_n)] - E[I(X_{k_0})]| \leq \varepsilon$

$$\limsup_{n \to \infty} P(I(X_n) \geq \lambda) = \limsup_{n \to \infty} E[I(X_n)] \leq \lim_{\lambda \to \infty} \frac{E[I(X_0)] + C}{\lambda} = 0$$

So an easy application of DCT gives $\exists E_y \in E$

$$x \geq x_0 \Rightarrow \sup_{n \geq k_0} \int_{x_n \geq -\varepsilon} I(x_n) \, dP \leq \varepsilon$$

As noted in the hint, by the result for $x_n \geq 0$ (shown in class)

$$\exists \lambda, \forall \theta, \lambda > \lambda, \Rightarrow \sup_{n \geq k_0} E[I(X_n)](x, \lambda) < \varepsilon.$$
So use (2) to see that for \( \lambda \geq \lambda_0 \) and \( k = k_0 \),

\[
\begin{align*}
\forall n \in \mathbb{N} & \quad E(\|X_n\|^2) \leq 3 + \frac{3}{1 + \|X_0\|^2} + E(\|X_0\| 2 \|X_n\|^2) \\
& \leq 32 \\
& \text{by (6)}.
\end{align*}
\]

By increasing \( \lambda \), we may also assume that

\[
\forall n < \infty \quad E(\|X_n\| 2 \|X_{n+1}\|) \leq 32 \quad (\lambda \geq \lambda_0),
\]

\[
\forall n \in \mathbb{N} \quad E(\|X_n\|^2) \leq 32 \quad \forall \lambda \geq \lambda_0
\]

\[
\exists \lambda_0 < \infty \quad \forall n \in \mathbb{N}
\]

\[
\Box
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