Math 546 Assignment 1 (due Thurs. Oct. 3)

1. If \( X_t, t \geq 0 \) and \( Y_t, t \geq 0 \) are a.s. right-continuous stochastic processes such that \( Y \) is a version of \( X \), prove that \( X \) and \( Y \) are indistinguishable.

2. Let \( (\mathcal{F}_t^0)_{t \geq 0} \) be a filtration on a complete probability space \((\Omega, \mathcal{F}, P)\) and let \( \mathcal{N} \) be the set of \( P \)-null sets in \( \mathcal{F} \). Show that \( \mathcal{F}_t = \overline{\mathcal{F}_t^0} \) is right-continuous (and hence satisfies the usual hypotheses).

    **Hint:** You may assume the standard measure theoretic result that \( A \in \mathcal{F}_t \) if and only if \( A = (B \cup N^1) - N^2 \) for some \( N^i \in \mathcal{N} \) and \( B \in \mathcal{F}_t^0 \).

3. Let \( (B_t, t \geq 0) \) be an \( (\mathcal{F}_t^0) \)-Brownian motion. Prove it is also an \( (\mathcal{F}_t) \)-Brownian motion where \( \mathcal{F}_t = \overline{\mathcal{F}_t^0} \).

    **Hint:** One approach is to let \( A \in \mathcal{F}_s (P(A) > 0) \) and find the (conditional on \( A \)) characteristic function (for \( t > s \geq 0 \) and \( \theta \in \mathbb{R}^d \)),

    \[ E(\exp(i\theta \cdot (B_t - B_s))|A). \]

4. If \( B \) is a standard one-dimensional \( (\mathcal{F}_t^0) \)-BM prove that \( M(t) = B(t)^2 - t \) is an a.s. continuous \( (\mathcal{F}_t^0) \)-martingale.

5. Law of the iterated logarithm–upper bound. If \( B \) is a standard one-dimensional Brownian motion, prove that

    \[ \limsup_{t \to 0^+} \frac{B_t}{\phi(t)} \leq 1 \text{ a.s.}, \]

    where \( \phi(t) = \sqrt{2t \log \log(1/t)} \) for \( t < e^{-1} \).

    **Hint.** Let \( M_t = \sup_{s \leq t} B_s \). Let \( r \) and \( c \) be real numbers such that \( 1 < 1/r < c^2 \). Bound the probabilities \( P(M_r/n > c\phi(r^n)) \) when \( n \to \infty \) and infer that a.s.,

    \[ \limsup_{t \to 0^+} \frac{B_t}{\phi(t)} \leq 1 \text{ a.s.} \]

6. [Not to hand in] Let \( \{X_n : n \in \mathbb{Z}_+\} \) be a reverse submartingale satisfying \( \inf_n E(X_n) > -\infty \). Prove that \( \{X_n\} \) is uniformly integrable.

    **Hint:** We proved in class that the result holds for the non-negative reverse submartingale \( X_n^+ \) and therefore that

    \[ \lim_{\lambda \to \infty} \sup_n E(X_n 1(X_n \geq \lambda)) dP = 0. \]

    Next, prove that for \( \lambda > 0, n > k \)

    \[ E(X_n 1(X_n \leq -\lambda)) \geq E(X_n) - E(X_k) + E(X_k 1(X_n \leq -\lambda)). \]