**Theorem 6.9 (Ginibre)**

1. A unique probability \( Q^x = \nu^x \) (Wiener measure) on \( (W_o, W, \mathbb{P}) \) exists.

2. For \( x \in \mathbb{R} \),
   \[
   \frac{dQ^x}{dx} = \exp \left\{ \int b(Y_s) \cdot dY_s - \frac{1}{2} \int \| b(Y_s) \|^2 ds \right\}
   \]
   under \( Q^x \).
   \[ B_t = Y_t - x - \int b(Y_s) ds \] is a standard \( \mathbb{W} \)-Brownian motion.

3. Hence on the set-up \( (W_o, W, \mathbb{P}) \), \( B, x \) \( Y \) is a solution of
   \[
   \frac{dX_t}{dt} = \nu_0 y_t \quad \forall t \in [0, T] \quad \text{and} \quad X_0 = x
   \]
4. Let \( X_t \) be a solution of \( (5.0.2) \), \( X_{t, b} \) on a set-up \( \mathbb{R}_+ \)
5. and define \( P^X \) on \( (W_o, W, \mathbb{P}) \) by
   \[
   P^X(A) = P(X_t \in A)
   \]
6. Then \( P^X = Q^x \). (So solutions to \( (5.0.2) \) are unique in \( \mathbb{W} \).)

**Proof**

1. If \( X \) solves \( (5.0.2) \), then
   \[
   B_t = Y_t - x - \int b(Y_s) ds \quad \text{is on} \quad (W_o) - B A, \text{and}\end{equation}
   \[
   \begin{aligned}
   \text{adapted} \quad & \text{on} \quad \mathbb{R}_+ - B A \\
   \text{Define} \quad & (b) \quad B_t = Y_t - x - \int b(Y_s) ds \quad \text{on} \quad \mathbb{W} = (W_o, W, \mathbb{P}) \quad (1.w, P^X, B_t)
   \end{aligned}
   \]

2. We know
   \[
   \text{} \quad P^X(e^{\mathbf{1} \mathbf{0} \cdot (B_t - B_0)} | X_0) = e^{-\|a\|^2 t \mathbf{1} \mathbf{0} \cdot (B_t - B_0)}
   \]
   \[
   \text{This implies} \quad \text{E} \mathbf{1} \mathbf{0} \cdot (B_t - B_0) = 0
   \]

3. This is an easy calculation, using finite dimensional events in \( W_o^0 \) of the form \( \frac{1}{3} \left( Y_{\alpha}, \ldots, Y_{\beta} \right) \in A \)

4. \( \mathbf{1} \mathbf{0} = \alpha \pm \beta \quad \varepsilon_{\gamma} = 0 \pm \varepsilon \)
\[(a) \Rightarrow B \text{ is } \mu(W_0) - \text{BM} \quad (W_0, \mu) \quad \text{under } P_x. \]

\[\Rightarrow B \text{ is } \mu(W_0) - \text{BM} \quad (W_0, \mu) \quad \text{under } P_x \quad \text{(NW1.43)}\]

Working under \(P_x\), let \(\tilde{\mathbf{M}}_t = -\int_0^t b(y_s) \cdot dB_s\).

\[
\mathbb{E}^{P_x}[\tilde{\mathbf{M}}_t^2] = \int_0^T \left[ \int_0^1 b(y_s) \right]^2 \, dt = \|b\|_2^2.
\]

By Prop 0.7, \(\tilde{\mathbf{M}}\) is a \((W_0)\) - martingale and so we may define a probability \(\tilde{P}^x\) on \((W_0, \mathcal{F}_T, \mathbb{P})\) by

\[
\frac{d\tilde{P}^x}{dP_x} = \mathbb{E}^{P_x}(\tilde{M}) > 0 \quad P_x \text{-a.s.}
\]

\[
\tilde{P}^x = P_x.
\]

By Girsanov's Thm, \(N_t = B_t^0 - \int_0^t [B_t^0, M_t^0] \, dB_t^0 \in \mathcal{M}_{0,1}(\tilde{P}_x)\)

\[
\text{with} \quad \tilde{N}_t = \frac{1}{2} \int_0^t [b(y_s), b(y_s)] \, dB_t^0.
\]

\[
[B_t^0, B_t^0] = \mathbb{E}^{\tilde{P}_x} \text{ under } \tilde{P}_x \text{ say}.
\]

By Lévy's Thm, \(N_t = (W_t^0) - \text{Br. Motion (standard)}\) under \(\tilde{P}_x\).

Use (3) in 10i to see that

\[
y_s = x + N_s \quad \text{is a } (W_t^0) - \text{Br. motion under } \tilde{P}_x
\]

\text{(of one-dimensional.)}

So \(\frac{d\tilde{P}^x}{dP_x} = \mathbb{E}^{P_x}(-\tilde{M}) > 0 \quad P_x \text{-a.s.}
\]

\[
dB_s/d\tilde{P}^x = \frac{1}{\mathbb{E}(-\tilde{M})} \quad \text{[as an easy check shows]}
\]

\[
= \exp\left\{ \frac{T}{2} \int_0^y b(y_s) \cdot dB_s + \frac{T}{2} \int_0^y \|b(y_s)\|^2 \, ds \right\}
\]

(by 10i)

\[
= \exp\left\{ \frac{T}{2} \int_0^y b(y_s) \cdot dB_s + \frac{1}{2} \int_0^T \|b(y_s)\|^2 \, ds \right\}
\]
\[ \frac{dP_x}{dp_x} = \exp \left( \sum_{b} \frac{1}{p_b} \right) \cdot \frac{dP_x}{dp_x} - \frac{1}{2} \int \frac{1}{p_1 p_2} \Pi_{2}^{\text{max}} d\Pi \]

\[ = \frac{dQ^x}{dp_x} \]  

(see (a)).

\[ P_x = Q^x \]