Girsanov's Thm. 1. \( V, \mathcal{B}, V \), \( \mathbb{P}, \mathbb{Q} \)

Notation: Let \( P, Q \) be probability on \( (\mathcal{F}, \mathcal{B}) \), and \( \mathcal{G} \) be a \( \mathcal{G}-\)filtered

\[
P = Q \text{ on } \mathcal{F} \Leftrightarrow P(\cdot|\mathcal{G}_t) = Q(\cdot|\mathcal{G}_t),
\]

\[
\left(\Leftrightarrow \forall \mathcal{G}_t \text{ P}()\mathcal{G}_t = Q()\mathcal{G}_t \right).
\]

\[
P = Q \text{ on } \mathcal{F}_t \text{ if } \forall \mathcal{G}_t \text{ P}()\mathcal{G}_t = Q()\mathcal{G}_t
\]

\[
\text{[P = Q on } \mathcal{F}_t \text{ is too strong for most interesting cases]}
\]

Thm. G.11. Assume \( Q \prec P \). Let \( Z_t = \frac{dQ}{dP} Y_t = \frac{dQ}{dP} S_t \).

(a) \( Z_t = \text{ martingale w.r.t. } P \) s.t. \( Z_t > 0 \) \( \forall t \geq 0 \) \( P \)-a.s., \( E_P(\mathbb{E}Z_T) = 1 \) \( \forall t \geq 0 \).

(b) \( N_t = \text{ local martingale w.r.t. } Q \) \( Z_t > 0 \) \( N \text{ is real } Q \text{-local} \).

(c) (Girsanov I) Assume \( Z \) is continuous. Then for any \( \mathcal{G}_t \),

\[
\exists M \in \mathcal{M}_{loc}(P) s.t. Z_t = Z_0 \mathbb{E}_t [M], \quad (\forall t \geq 0).
\]

(i) \( Y \in \mathcal{M}_{loc}(P) \Leftrightarrow Y \in [Y_t, M] \in \mathcal{M}_{loc}(Q) \).

(ii) \( L(Y_t) = 0 \mathcal{B}(\mathcal{F}) \).

(Recall of H.W.): (a) \( Z_t \geq 0 \) \( \text{super-martingale}, \mathcal{F}_t \mathcal{F}_t \).

(\( Z_t = \mathbb{E}_t [Y_T] \geq \mathbb{E}_t [Y_T] = \mathbb{E}_t [Y_T] \).

(c) \( Z_t \) holds without continuity assumptions on \( Z \) or \( M \) and there \( M \)

(a) \( Z_t \) holds with continuity assumptions on \( Z \) or \( M \) and there \( M \)

\( \text{aversion which holds when } \mathbb{F}_t = \mathcal{F}_t \).

See VII.48-50 in Dellacherie—Heyer, Prob. and Potentials B.

Recall (H.W.): (a) \( Z_t \geq 0 \) \( \text{super-martingale}, \mathcal{F}_t \mathcal{F}_t \).

(\( Z_t = \mathbb{E}_t [Y_T] \geq \mathbb{E}_t [Y_T] \)).
\( P(2) \) \( \leq \lambda \) \( \text{A} \in \mathcal{Y}_\lambda \)

\[
\sum_{Z \in \mathcal{Y}_\lambda} dP = G(\lambda) = \int_{\mathcal{A}} Z dP = \sum_{Z \in \mathcal{A}} dP \Rightarrow Z \text{ is odd} \)-mean.
\]

\[
G(\lambda Z_\lambda = 0) = \sum_{Z \in \mathcal{A}} Z dP = 0
\]

\[
\lambda = \lambda Z_\lambda = 0
\]

\[
\therefore P(2) = 0 \quad \Rightarrow G(\lambda Z_\lambda = 0) = P(2).
\]

\[
Z_\lambda > 0 \quad \forall \lambda > 0
\]

\[
\therefore Z_\lambda > 0 \quad \text{for all } \lambda > 0, \quad \text{a.s.}
\]

\[
\therefore Z_\lambda > 0 \quad \forall \lambda \geq 0, \text{ a.s.} \quad \text{by } (2)
\]

Finally \( \sum_{Z \in \mathcal{Y}_\lambda} dP = G(\lambda Z) \rightarrow 1 \quad \forall \lambda \geq 0 \).
(1) Let \( N \) be \( \mathcal{L}_\mu \)-adapted cadlag, \( s \leq b \) and \( \mathcal{A} \subseteq \mathcal{F}_s \). Consider the inequalities:

\[ \int_A N(x) \, dq = \int_A N(x) \, d\mu \]

\[ \int_A N(x) Z(x) \, dp = \int_A N(x) Z(x) \, d\mu \]

By definition, \( Z(x) \) and \( Z(y) \) are \( \mathcal{L}_\mu \)-measurable, \( \mathcal{F}_s \)-measurable.

(2) \( N \) is \( \mathcal{F}_s \)-measurable, \( \mathcal{L}_\mu \)-measurable. By the above, \( N \) is \( \mathcal{F}_s \)-measurable.

Now interchange \( P \) and \( Q \) in previous result: \( N \in \mathcal{M}_{\mathcal{L}_\mu}(\mathcal{P}) \).
(c)(i) \( Y - (-1, M) \in c \cdot M_0, o_c (P) \)
\( \iff Z = Z_1 \cdot (Y - (-1, M)) + [Z, Y] \in c \cdot M_0, o_c (P) \) \( \text{(ZBP)} \)

\( \iff \frac{Z}{M_0, b_c (P)} \)

(3) \( \iff Z \cdot Y = \frac{1}{\varepsilon} [Z, \varepsilon \cdot (Y - (-1, M))] + [Z, Y] \in c \cdot M_0, o_c (P) \)

\( \text{by Thm 5.13, } \frac{Z}{\varepsilon} = Z_1 + \frac{1}{\varepsilon} [Z, \varepsilon \cdot (Y - (-1, M))] \), so \( [Z, Y] = \frac{1}{\varepsilon} [Z, d \cdot (M, Y)] \).

\( \therefore (3) \iff Z \cdot Y \in c \cdot M_0, o_c (P) \).

Hence to complete the proof we need to show

(4) \( Z \cdot Y \in c \cdot M_0, o_c (P) \Rightarrow Y \in c \cdot M_0, o_c (P) \)

\( \implies \) immediate.

(4) \( \text{Note } \frac{1}{\varepsilon} \in \text{ib } P \implies Z \gg 0 \implies z \gg 0 \text{ a.s.} \)

\( \therefore Y = \frac{1}{\varepsilon} [Z, Y] = \frac{1}{\varepsilon} (Z \cdot Y) \in c \cdot M_0, o_c (P) \)

\( \therefore Z \cdot Y \in c \cdot M_0, o_c (P) \).

This shows (4) and completes the proof.

(ii) Let \( X = X_0 + N \), \( X \in 9 (P) \), \( N \in c \cdot M_0, o_c (P), V \in c \cdot V (P) \)

\( = X_0 + [N - (1, V, M)] + [V + (1, N, M)] \)

\( = X_0 + [N - (1, V, M)] + [V + (1, M, N)] \text{ (CF, FV, P, CF, FV, A)} \)

\( \therefore X \in c \cdot M_0 (A) \). \( \therefore c \cdot D (P) \subset c \cdot D (b) \).

Now interchange P and Q to see \( c \cdot D (P) = c \cdot D (b) \).
To go the other way and define \( L(X) = E_k(M) \), we need (at the least by theorem), \( E_k(M) \) is an \( L \)-mapping of \( \mathbb{R} \). Here is a convenient sufficient condition.

Theorem 13 (Nonco) If \( M \in \mathcal{M} \), satisfies \( E_k^{p, \lambda}(L) \geq A \; \forall \lambda > 0 \)

For some \( p \geq \frac{1}{2} \), then \( E_k(M) \) is an \( (L) \)-mapping of \( \mathbb{R} \).

Remark: Also holds if \( p = \frac{1}{2} \) — see Ikeda + Wakonabe Ch III Ths 3.

[Pl on web.]

Remark: Let \( p \approx q \), \( X \in \mathcal{B}(L) \) .

\[ T_n \rightarrow L \; \Leftrightarrow \; T_n \rightarrow p \rightarrow q \; \text{as in proof of local mapping case of Cor.} \]

Claim: \( H^p X = H^q X \).

[Proof]

Prop: \( H^p X \). Now apply Lien zer Thms 4, 32. to show \( H^p X \) defines a map of \( H_n \rightarrow H \), \( \text{s.t.} \; H_n \left( \psi \right) = K_n \alpha \) \( c \in L \).

\[ \Rightarrow \text{by } \alpha, \beta \in \mathcal{R} \left( H_n, \varphi \right) \quad H_n \left( \psi \right) 
\]

We know \( \left( H_n, \psi \right) \rightarrow \mathbb{R} \). \[ \Rightarrow \text{by } \alpha, \beta \in \mathcal{R} \left( H_n, \varphi \right) \quad H_n \left( \psi \right) 
\]

Wolog \( \psi \left( 1, H_n, \varphi \right) \rightarrow \mathbb{R} \; \text{as in lemma} \)

[This contradicts (4).]
We will work on the canonical space of pairs \((W^o, W^d)\) with \(\gamma_k(w) = w_k\). Let \(P^x\) be Wiener measure on \(\text{stirling}_d\) (a probability on \((W^o, W^d)\)).

Assume \(b : \mathbb{R}^d \to \mathbb{R}^d\) is bounded Borel measurable.

We will use Girsanov to construct solutions \(\tilde{\omega}_t\) and establish uniqueness in law, i.e.,

\[
\text{(5.0.2)} \quad \tilde{\omega}_t = x + B_t + \int_0^t b(\tilde{\omega}_s) \, ds
\]

**Theorem (Girsanov).**

(i) There exists a unique probability \(Q^x\) on \((W^o, W^d)\) s.t.

\[
\lim_{\Delta \to 0} \frac{dQ^x}{dP^x} |_{W^d_t} = \exp \left\{ \frac{1}{2} \int_0^t b(\tilde{\omega}_s) \cdot dB_s - \frac{1}{2} \Delta \int_0^t ||b(\tilde{\omega}_s)||^2 \, ds \right\}
\]

Moreover, \(Q^x \sim P^x\) and under \(Q^x\), \(B_t = \tilde{\omega}_t - x - \frac{1}{2} \int_0^t b(\tilde{\omega}_s) \, ds\) is a standard \((W^o_0)\)-Brownian motion. Therefore \(\tilde{\omega}\) is a solution of \((5.0.2)\) \(t \mapsto b\) under \(Q^x\).

(ii) Let \(x\) be a solution of \((5.0.2)\) \(t \mapsto x + \beta_t + \int_0^t b(\tilde{\omega}_s) \, ds\) \((\tilde{x} = (\tilde{x}_1, \tilde{x}_2, P, \beta, \tilde{\omega} = x))\)

on a set-up \(\mathcal{F}\), and let \(P_x\) be the law of \(\tilde{x}\) on \((W^o, W^d)\). Then \(P_x = Q^x\) and so "solutions to \((5.0.2)\) \(t \mapsto b\) are unique in law".

**Proof.** (i) \(P_x = \frac{1}{2} \int_0^t b(\tilde{\omega}_s) \cdot dB_s \) is a Markov process under \(P^x\), \( \lim_{\Delta \to 0} \Delta \int_0^t ||b(\tilde{\omega}_s)||^2 \, ds = \frac{1}{2} \int_0^t ||b(\tilde{\omega}_s)||^2 \, ds \)

and so by Novikov's Theorem \(Z_t = E_{\tilde{\omega}_0}^x\) is a \((W^o)\)-martingale.

Let \(A \in \mathcal{F} \cup W^o_0 = \mathcal{W}_0\) a field. If \(A \in \mathcal{W}_0\), \(s \leq t\), then

\[
\int_A Z_t \, dP^x = \int_A Z_s \, dP^x \quad (Z_t \text{ is a martingale})
\]

and so we may define \(Q^x(A) = \int_A Z_t \, dP^x \) (indifferent choice).

and so
\[ \frac{\partial G^x}{\partial t} \bigg|_{(t=0)} = Z \] as in the statement.

\[ Q^x \text{ is finitely additive on } \mathcal{U}. \] (\( Q^x \) is countably additive on each \( \mathcal{W} \).

Arguing as in the proof of the Kolmogorov Batorin, one sees that \( Q^x \) is countably additive on \( \mathcal{W} \).

[PL] Let \( \mathbf{A} \in \mathcal{W} \), \( \mathbf{A}_n \uparrow \mathbf{A} \). It suffices to show \( Q^x(A_n \uparrow \mathbf{A}) \to 0 \).

Suppose \( \epsilon > 0 \) s.t. \( Q^x(A_n \uparrow \mathbf{A}) > \epsilon \). Let \( \mathbf{A}_n \in \mathcal{W}_n \) where \( \mathfrak{w} \mathbf{A}_n \to \mathbf{A} \) as \( n \to \infty \). As \( C(L^0, \mathbb{R}) \) is Polish and \( Q^x|_{\mathcal{W}_n} \) is a probability, by inner regularity \( \exists K_n \text{ compact in } C(L^0, \mathbb{R}) \) s.t. \( B_n = \{ w \in \mathcal{W}_n \mid C_{K_n} \subseteq C_{K_n} \} \), \( Q^x (A_n \uparrow \mathbf{A}) \leq \epsilon \).

\[ Q^x(\bigcap_{n=1}^\infty B_n) \leq \frac{n}{\epsilon} Q^x(A_n \uparrow \mathbf{A}) < \epsilon, \text{ and } Q^x(A_n \uparrow \mathbf{A}) \to 0. \]

\[ \bigcap_{n=1}^\infty B_n \cap C_{K_n} = \emptyset, \text{ a contradiction.} \]

By the Carathéodory Batorin Thm \( Q^x \) has a unique extension \( \{ w^0, w^d \} \to \) (uniformly in \( C(L^0, \mathbb{R}) \)).

So \( \exists w^0 \in \mathcal{W}_n \cap \mathcal{W}_{\mathbf{A}} \cap C_{K_n} \), \( \sup n \{ w^0 \} \), \( \exists w \in \mathcal{W}_n \cap C_{K_n} \), \( \forall i \in \mathcal{A} \), \( w \in \bigcap_{n=1}^\infty B_i \), \( \mathbf{A}_n \), \( \mathbf{A}_n \) is a contradiction.

By the Carathéodory Batorin Thm \( Q^x \) has a unique extension \( \{ w^0, w^d \} \to \) (uniformly in \( C(L^0, \mathbb{R}) \)).

Clearly \( d Q^x|_{\mathcal{W}_n} = \mathcal{Z} > 0 \). \( p^2 \downarrow 0 \).

\( Q^x = p^x \).

Uniqueness \( x_1 \) is clear: \( Q^x|_{\mathcal{W}_n} \) is uniquely defined (so apply Carathéodory).

Under \( p^2 \), \( \gamma_i - x_i \in \mathcal{M}_{\mathcal{W}_n} \) and so by Girsanov's theorem \( \mathbf{B}_n = \gamma_i - x_i - \frac{\partial G^x}{\partial t} \bigg|_{(t=0)} \) is a controllable martingale under \( Q^x \).

\[ \mathbf{B}_n = \gamma_i - x_i - \frac{\partial G^x}{\partial t} \bigg|_{(t=0)} \]
Consider \( \{ \mathbf{B}^i, \mathbf{B}^j \} \mid \omega = \mathbf{L}^i \mathbf{j} \mid \omega = \delta_{ij} \) under \( \mathbb{P}^x \) or \( \mathbb{Q}^x \).

By Lévy's Thm, \( B \) is a \( \mathcal{L}(\mathbb{R})^d \) - Br. motion (standard) under \( \mathbb{Q}^x \).

Clearly, \( Y \) is a solution of \( L^0 \mathcal{D} \) under \( \mathbb{Q}^x \) by (i).

(b) Assume \( X \) solves \( L^0 \mathcal{D} \), on \( \mathcal{F} \) and \( Y \) - Br. motion \( \beta \) \( \beta \mid \mathcal{F}_t = X - \int_0^t \mathbf{b}(x, \mathcal{F}_s) \, ds \), \( \beta \) is \( \mathcal{F}^X \) - adapted; i.e., \( \mathbb{P}^x \) - Br. motion.

Let \( P_x (A) = P (X \in A) \) for \( A \in \mathcal{F}^X \).

Work on \( \tilde{\mathcal{F}} = \{ \mathcal{L}(\mathbb{R})^d, \mathcal{L}(\mathbb{R})^d, \mathcal{L}(\mathbb{R})^d \}, P_x \) and \( \mathcal{F} = X \).

(i) \( \beta \mid \mathcal{F}_t = \mathbf{X} - \int_0^t \mathbf{b}(x, \mathcal{F}_s) \, ds \).

\[ \mathbb{E} (e^{i \mathbf{b}(x, \mathcal{F}_s)} \mid \mathcal{F}_s) = e^{i \mathbf{b} \cdot \mathbf{X} - \frac{1}{2} \mathbf{b} \cdot \mathbf{b} \cdot \mathcal{F}_s} \quad \text{trivial}. \]

\[ \mathbb{E} (e^{i \mathbf{b}(x, \mathcal{F}_s)} \mid \mathcal{F}_s) = e^{- \frac{1}{2} \mathbf{b} \cdot \mathbf{b} \cdot \mathcal{F}_s} \text{ mod.} \]

So \( B \) is a \( \mathcal{L}(\mathbb{R})^d \) - Br. motion (standard) under \( P_x \).

Now define \( \frac{dP_x}{dP} = e^{i \mathbf{b} \cdot \mathbf{X} - \frac{1}{2} \mathbf{b} \cdot \mathbf{b} \cdot \mathcal{F}_s} \) for \( \mathcal{F} \).

\[ \frac{dP_x}{dP} = e^{\mathbb{E}(\mathbb{M}^i \mid \mathcal{F}_s)} \]

where \( \mathbb{M} \mid \mathcal{F}_s = - \frac{1}{2} \mathbf{b} \cdot \mathbf{b} \cdot \mathcal{F}_s \).

Using Novikov's Thm, as in (a), \( E \mathbb{M}^i \) is a \( \mathcal{L}(\mathbb{R})^d \) - martingale.

As in (a), \( P_x \) extends to all prob. on \( \mathcal{L}(\mathbb{R})^d \).

By Girsanov Thm, \( N_t = \mathbf{B}^i + \int_0^t \mathbf{b}(x, \mathcal{F}_s) \, ds \) is a \( \mathcal{L}(\mathbb{R})^d \) - local martingale under \( P_x \).

As \( \mathbb{E}(\mathbf{B}^i, \mathbb{B}^j \mid \mathcal{F}_s) = \mathbb{E}(\mathbf{B}^i, \mathbb{B}^j \mid \mathcal{F}_s) = \delta_{ij} \), by Lévy, \( N \) is a \( \mathcal{L}(\mathbb{R})^d \) - Br. motion under \( P_x \), Compare (a) and (b) gives.

\[ Y_t = x + N_t \] is a Br. motion starting at \( x \) under \( P_x \).

\( P_x \) is Wiener measure starting at \( x \) (as notation suggests).
\[
\frac{dP_x}{dP^n|w_v^\nu} = \frac{1}{\Sigma(w_v^\nu)} = \exp\left\{ \int_0^1 b(y_s^\nu) - dB_s + \frac{1}{2} \int_0^1 \left\| b_1 Y_s \right\|^2 ds \right\}
\]

\[
= \exp\left\{ \int_0^1 b(y_s^\nu) \cdot [dy_s^\nu - b(y_s^\nu) ds] + \frac{1}{2} \int_0^1 \left\| b_1 Y_s \right\|^2 ds \right\}
\]

\[
= \exp\left\{ \int_0^1 b(y_s^\nu) \cdot dy_s^\nu - \frac{1}{2} \int_0^1 \left\| b_1 Y_s \right\|^2 ds \right\}
\]

Comparing with

\[
= \frac{dQ^x}{dP^n|w_v^\nu}
\]

\[\therefore P_x = Q^x, \quad \square\]