Math 421/510 Assignment 4 (due Thurs. March 20)

1. Let $X$ be a compact Hausdorff space and let $f_n \in C(X, \mathbb{R})$ decrease pointwise to 0. Prove that $\{f_n\}$ converges uniformly to 0.

2. p. 170 #43, #46, #48

3. p. 170 #49 (We did the weakly open case of (a) in class).

4. Let $\mathcal{X}$ be a n.l.s. Prove that any weakly compact set in $\mathcal{X}$ is bounded in norm.

5. p. 178 #63(b)

6. Topological Completeness.
   Let $X = (0, 1]$ and define a pair of metrics on $X$ by $d_o(x, y) = |x - y|$ and $d(x, y) = |x^{-1} - y^{-1}|$. Convince yourself, but not the grader, that this are indeed metrics. Clearly the $d_o$-topology on $X$ is the usual (subspace) topology $X$ inherits from the real line.
   (a) Show that the $d$-topology coincides with the $d_o$-topology.
   (b) Show that $(X, d_o)$ is not complete but that $(X, d)$ is complete.
   This shows that completeness is not a topological property but rather a property of the particular metric. A topological space $X$ is called topologically complete iff it is metrizable by a complete metric.
   (c) Let $(X, d)$ be a complete metric space and let $U \subset X$ be open. Prove that $U$ is topologically complete. (Consider $\rho(x, y) = d(x, y) + |f(x) - f(y)|$, where $f(x) = 1/d(x, U^c)$ and $d(x, U^c) = \inf\{d(x, y) : y \in U^c\}$.)

7. p. 187 #7

8. p. 192 #22