Math 421/510 Assignment 3 (due Tues. March 4)

1. p. 165 #38

For all \( x, y \in X \), and \( \lambda \in K \), \( T(\lambda x + y) = \lim_n T_n(\lambda x + y) = \lim_n \lambda T_n(x) + T_n(y) = \lambda Tx + Ty \). So \( T \) is linear. For each \( x \in X \), \( \sup_n \|T_n x\| < \infty \) since \( \{T_n x\} \) converges.

Therefore by the Uniform Boundedness Principle \( \sup_n \|T_n\| = C < \infty \). Therefore \( \|Tx\| = \lim_n \|T_n x\| \leq C \|x\| \). Therefore \( T \) is bounded.

2. p. 118 #2

If \( x \neq y \) then \( \{x\}^c \) is an open neighbourhood of \( y \) not containing \( x \). Hence \( X \) is \( T_1 \).

On the other hand if \( U_x \) and \( U_y \) are open neighbourhoods of \( x \) and \( y \), respectively, then \( (U_x \cap U_y)^c = U_x^c \cup U_y^c \) is finite and hence \( U_x \cap U_y \) is non-empty since \( X \) is infinite. This shows \( X \) is not \( T_2 \).

Suppose \( X \) has a countable neighbourhood base, \( \{U_n\} \) at \( x_0 \). As \( X \) is \( T_1 \), it follows that \( \cap_n U_n = \{x_0\} \). Therefore \( X - \{x_0\} = (\cap_n U_n)^c = \cup_n U_n^c \) is a countable union of finite sets and hence \( X \) is countable.

Suppose \( X \) is countable. Then \( \mathcal{T} = \{\emptyset\} \cup \cup_{n=1}^{\infty} \{F^c : F \subset X, \text{card}(F) = n\} \) is a countable union of countable sets and so is countable. In particular \( X \) is clearly first countable.

3. If \( \mathbb{R} \) is given the cofinite topology show that \( \mathbb{R} \) is separable but not first countable. (Clearly you should use Q. 2 for part of this).

Since \( \mathbb{R} \) is uncountable, Q.2 shows that it is not first countable in the cofinite topology. If \( U \) is any non-empty open set the fact that \( U^c \) is finite means the \( U^c \) cannot contain \( \mathbb{Q} \) and so \( \mathbb{Q} \cap U \) is non-empty. Therefore \( \mathbb{Q} \) is a countable dense set and so \( \mathbb{R} \) is separable.

4. p. 118 #5

Let \( \{x_n\} \) be a countable dense set in the separable metric space \( X \) and \( \mathcal{B} = \{B(x_n, r) : n \in \mathbb{N}, r \in \mathbb{Q}, r > 0\} \). Clearly \( \mathcal{B} \) is countable and we now show it is a base for \( X \). Let \( x \in U \) for some open \( U \). Choose \( r > 0 \) rational so that \( B(x, r) \subset U \). Choose \( x_n \in B(x, r/2) \). Then \( V = B(x_n, r/2) \in \mathcal{B} \) and \( x \in V \subset U \), the latter inclusion by the triangle inequality.

This shows \( \mathcal{B} \) is a neighbourhood base at the arbitrary point \( x \) and hence is a countable base for \( X \).

5. p. 123 #17

Assume for all \( x \neq y \) there is an \( f \in \mathcal{F} \) so that \( f(x) \neq f(y) \). Fix \( x \neq y \) and choose \( f \) as above. As \( \mathbb{R} \) is Hausdorff we may choose disjoint open nbhd.’s, \( U_x \) and \( U_y \), of \( f(x) \) and \( f(y) \) in \( \mathbb{R} \). Then \( f^{-1}(U_x) \) and \( f^{-1}(U_y) \) are disjoint open sets in \( X \) containing \( x \) and \( y \), respectively, and so \( X \) is Hausdorff.

Assume \( X \) is Hausdorff. Let \( x \neq y \) in \( X \). Then there is a basic open set of the form \( \cap_i f_i^{-1}(U_i) \) containing \( x \) but not \( y \) where \( f_i \in \mathcal{F} \) and \( U_i \) is open in \( \mathbb{R} \). Therefore for some \( i_0 \) we have \( f_{i_0}(y) \notin U_{i_0} \) but \( f_{i_0}(x) \in U_{i_0} \). This implies \( f_{i_0}(x) \neq f_{i_0}(y) \) and we are done.

6. p. 127 #32, 33, 34

#32. Assume \( X \) is not \( T_2 \). Let \( x \neq y \) be such that any open neighbourhoods of \( x \) and \( y \) intersect. Let \( \mathcal{N}_x \) and \( \mathcal{N}_y \) be the sets of open nbhd’s of \( x \) and \( y \), respectively. We make \( I = \mathbb{N}_x \times \mathbb{N}_y \) into a directed set by defining \( (U_x, U_y) \geq (V_x, V_y) \) iff \( U_x \subset V_x \) and
$U_y \subset V_y$. Define $x(U_x, U_y)$ to be a point in $U_x \cap U_y$, where $(U_x, U_y) \in I$. For any $U \in \mathcal{N}_x$ let $\alpha_0 = (U, V)$ where $V$ is some set in $\mathcal{N}_y$. If $(U_x, U_y) \geq (U, V)$, then $x(U_x, U_y) \in U_x \subset U$ and so $\lim_i x_i = x$. By symmetry we also get $\lim_i x_i = y$.

Assume $X$ is $T_2$. Assume $\lim_i x_i = x$ and $y \neq x$. Choose an open nbhd $U$ of $x$ and a disjoint open nbhd $V$ of $y$. Then $x_i \in U$ eventually and so $x_i \not\in V$ eventually. The latter means $\{x_i\}$ cannot converge to $y$. Hence a net can converge to at most one point.

#33. Assume $x \in \cap_\alpha \bar{E}_\alpha$. Let $U$ be an open nbhd of $x$ and $\alpha \in A$. Then $x \in \bar{E}_\alpha$ implies $U \cap E_\alpha$ is non-empty and so there is an $\alpha' \geq \alpha$ so that $x_{\alpha'} \in U$. This shows that $x_{\alpha}$ is in $U$ frequently and so $x$ is a limit point of $\{x_\alpha\}$.

Assume $x$ is a limit point of $\{x_\alpha\}$. Let $\alpha \in A$. If $U$ is an open nbhd of $x$, there is an $\alpha' \geq \alpha$ so that $x_{\alpha'} \in U$. This means that $E_\alpha \cap U$ is non-empty and so $x \in \bar{E}_\alpha$ (otherwise we could take $U = \bar{E}_\alpha^c$ in the above and obtain a contradiction). Therefore $x \in \cap_\alpha \bar{E}_\alpha$.

#34. Assume $\lim_\alpha x_\alpha = x$. If $f \in \mathcal{F}$, then $f$ is continuous in the weak-$\mathcal{F}$ topology and so $\lim f(x_\alpha) = f(x)$.

Assume $\lim f(x_\alpha) = f(x)$ for all $f \in \mathcal{F}$. Consider the basic neighbourhood of $x$, $U = \bigcap_{j=1}^n f_j^{-1}(V_j)$, where $f_j \in \mathcal{F}$ and $V_j$ is open in the range space of $f_j$. By hypothesis and the fact that $A$ is directed there is an $\alpha_0 \in A$ so that $\alpha \geq \alpha_0$ implies that $f_j(x_\alpha) \in V_j$ for all $j = 1, \ldots, n$. This means that $x_{\alpha} \in U$ for $\alpha \geq \alpha_0$ and so $\lim x_\alpha = x$. 