Proof of Transience of Simple Symmetric RW in $Z^d$.

$X_n = \sum_{j=1}^{2^d} Y_j$. Each of the steps $Y_j$ heads $E$, $W$, $N$, $S$, Up or Down.

$X_{2n} = 0$ iff $\#$steps $E = \#$steps $W = i$, $\#$steps $N = \#$steps $S = j$ and $\#$steps $U = \#$steps $D = k$.

Calculating this latter probability using the multinomial disribution gives:

$$p_n(X_{2n}=0) = \sum_{\{i,j,k\} : i+j+k=n} \binom{2n}{i,j,k} \left(\frac{1}{6}\right)^{2n}$$

$$= \sum_{\{i,j,k\} : i+j+k=n} \frac{2n!}{i!j!k!} \left(\frac{1}{6}\right)^{2n}$$

$$= \binom{2n}{n} \left(\frac{1}{2}\right)^{2n} \sum_{\{i,j,k\} : i+j+k=n} \left(\frac{n}{i,j,k}\right)^{2n}$$

$$\sum_{\{i,j,k\} : i+j+k=n} \leq p_n(1,1,0) = 1$$

as these are the multinomial probabilities with $n$ trials and 3 equally likely outcomes.

Consider minimizing $i!j!k!$ for $i+j+k=n$.

If say $i < \left\lfloor \frac{n}{3} \right\rfloor$ then $j$ (say) must be $> \left\lfloor \frac{n}{3} \right\rfloor$ and increasing $i$ by 1 and decreasing $j$ by 1 will decrease $i!j!k!$, by a multiplicative factor of $i/j$.

Hence the minimum is attained when $i,j,k \geq \left\lfloor \frac{n}{3} \right\rfloor$ and similarly $i,j,k \leq \left\lceil \frac{n}{3} \right\rceil$. 
So if \( \frac{1}{3} S_{n,3} \) is a state on \( \mathbb{Z} \), \( \frac{1}{2} \leq l_n, j_n, k_n \leq \frac{4}{3} \), \( \eta + j_n + k_n = n \) we have

\[
P_{\hat{S}} (X_{3n} = 0) \leq P(S_{2n} = 0) \frac{n!}{\eta! \mu! \nu!} \frac{1}{\eta^n \mu^n \nu^n} \frac{1}{(\eta + \mu + \nu)^n}
\]

\[
\leq \frac{e}{\eta^n \mu^n \nu^n} \frac{1}{(\eta + \mu + \nu)^n} \frac{1}{\eta^n \mu^n \nu^n} \frac{1}{(\eta + \mu + \nu)^n}
\]

(BY Stirling and \( d=1 \) calculation)

\[
\leq \frac{C'}{\left( \frac{n}{3} - 1 \right)^n} \left( \frac{n}{3} - 1 \right)^{3/2} \]

\[
\leq \frac{C''}{n^{3/2}} \left( \frac{n}{n-3} \right)^n \leq \frac{C_0}{n^{3/2}}
\]

\[
\frac{8}{\pi} \frac{P_{\hat{S}} (X_{3n} = 0)}{n^4} \leq \frac{8}{\pi} \frac{C_0}{n^{3/2}} < \infty
\]

As \( P_{\hat{S}} (X_{3n+1} = 0) \), this proves \( \hat{S} \) is transient and so any state in \( \mathbb{Z}^3 \) is transient.