5. TRUE OR FALSE. If True, provide a proof. If False, provide and justify a counterexample.

6 marks (a) Any finite state Markov Chain has a stationary initial distribution.

TRUE

Since $S$ is finite closed, $s_r$ is recurrent. Let $T_i = \inf\{ n \geq 0 | X_n = j \}$.

$$M_i(j) = E_i\{\frac{1}{T_i} 1_{X_n=j} \}$$

is a s.i. measure.

$$M_i(j) = \sum_{n=0}^{\infty} M_i(j) < \infty.$$

$$\frac{M_i}{M_i(j)}$$

is a s.i. d.

5 marks (b) An irreducible transient Markov Chain does not have a stationary measure.

FALSE

Let $(X_n)$ be SSRW in $\mathbb{Z}^3$. Then $(X_n)$ is transient (Polya's thm) but $\mu_{ij} = 2d$ $\forall j \in \mathbb{Z}^3$ is a s.i.m.

Here $\mu_{ij} = \# \{ \text{neighbours of } i \}$ $\forall j \in \mathbb{Z}^3$. 
TRUE OR FALSE

6 marks  (c) If $(X_n)$ and $(Y_n)$ are both $(F_n)$-submartingales, then so is $Z_n = \max(X_n, Y_n)$.

TRUE

1. $E(Z_{n+1}|Y_n) \geq E(X_{n+1}|Y_n) \geq X_n$ a.s.
2. $E(Z_{n+1}|Y_n) \geq E(Y_{n+1}|Y_n) \geq Y_n$ a.s.

$1+2 \Rightarrow E(Z_{n+1}|Y_n) \geq \max(X_n, Y_n) = Z_n$ a.s.

Other properties (adaptedness, integrability) are obvious so $(Z_n)$ is an $(Y_n)$-submartingale.
6. Let $G$ denote the complete undirected graph on $N$ vertices, where $N \geq 2$. That is, the vertex set is $S = \{1, 2, \ldots, N\}$ and there is an edge connecting any pair of distinct vertices. Let $(X_n)$ denote random walk on $G$ starting at $i$ under the probability $p_i$. For $i \in S$, let $T_i = \min\{n \geq 1 : X_n = i\}$.

(a) For any $i \in S$, find $E_i(T_i)$ and justify your answer.

For $G$, $p(i) = \# \{j : i \sim j\} = N - 1 \quad \forall i \in S$.
$p(i) = \frac{\sum_{j \neq i} p(i)}{p(i)} = N \frac{N - 1}{N}$.

$\nu(i) = p(i) = \frac{N - 1}{N}$.

$E_i(T_i) = \frac{1}{\nu(i)} = N \quad \forall i$.

(Recall $\nu(i) = \frac{1}{p(i)}$ is the unique a.i.d. for our PR class $S$)

(b) For each distinct $i, j$ in $S$ find $E_i(T_j)$ and justify your answer.

\text{Method 1:} By symmetry $E_i(T_j) = N$, independent of $j \neq i$.

(Re-label $j_1 \neq j, j_2 \neq j$, and dynamics i.e. $P$ is the same)

$N = E_i(T_j) = E_i \left( \sum_{n=1}^{\infty} 1_{T_j \leq n} x_{T_j} x_{T_{j+1}} \ldots \right)$.

$= 1 + E_i \left( E_i(T_j x_{T_j} x_{T_{j+1}} \ldots ) 1_{T_j < \infty} \right)$.

$= 1 + E_i \left( E_x(T_{T_j}) \right)$.

$= 1 + \sum_{j \neq i} \frac{1}{N - 1} E_i(T_j) = 1 + \left( \frac{N - 1}{N - 1} \right) N - 1$.

$E_i = N - 1$

\text{Method 2:} $P_j(T_j > n) = E_i \left( P\left( x_n \neq j \quad \forall \quad k \leq n - 1 \right) \right)$.

$= E_i \left( 1_{x_n \neq j} \quad \forall \quad k \leq n - 1 \right) P_{x_{n-1}}(x_n \neq j)$.

$= E_i \left( 1_{T_j > n-1} \right) \left[ 1 - \frac{1}{N - 1} \right]$.

$P_j(T_j > n) = \left( 1 - \frac{1}{N - 1} \right)^n \Rightarrow T_j \text{ is geometric } p = \frac{1}{N - 1}$.
7. Let \( \{X_n : n \in \mathbb{N}\} \) be independent r.v.'s with mean zero and such that \( |X_n| \leq B \) for all \( n \) and some real number \( B \). Let \( \sigma_k^2 \) denote the variance of \( X_k \).

(a) Prove that \( S_n = \sum_{k=1}^{n} X_k \) and \( M_n = S_n^2 - \sum_{k=1}^{n} \sigma_k^2 \) are martingales.

\[
\mathbb{E}\left(S_{n+1} | Y_n^X\right) = \mathbb{E}\left(S_n + X_{n+1} | Y_n^X\right) = S_n + \mathbb{E}\left(X_{n+1}\right) = S_n,
\]
\[
\mathbb{E}\left(M_{n+1} - M_n | Y_n^X\right) = \mathbb{E}\left(S_{n+1}^2 - S_n^2 | Y_n^X\right) - \sigma_{n+1}^2
\]
\[
= \mathbb{E}\left(2S_nX_{n+1} + X_{n+1}^2 | Y_n^X\right) - \sigma_{n+1}^2
\]
\[
= 2S_n\mathbb{E}\left(X_{n+1} | Y_n^X\right) + \mathbb{E}\left(X_{n+1}^2 | Y_n^X\right) - \sigma_{n+1}^2
\]
\[
= 0.
\]

As \( |X_n| \leq B \), integrability is obvious, as is adaptedness.
\( \therefore \) \( \{S_n\} \) and \( \{M_n\} \) are martingales.

(b) If \( \sum_{k=1}^{\infty} \sigma_k^2 < \infty \), prove that \( \sum_{k=1}^{\infty} X_k \) converges a.s., and \( \{S_n\} \) also converges in \( L^1 \).

\[
\mathbb{E}\left(S_n^2\right) = \sum_{k=1}^{n} \sigma_k^2 \leq \sum_{k=1}^{\infty} \sigma_k^2 < \infty
\]

\( \therefore \) \( \{S_n\} \) is an \( L^2 \)-bounded martingale.

\( \therefore \) \( \{S_n\} \) is a u.i. martingale. \( \therefore \) \( L^1 \)-bounded.

\( \therefore \) By Mart. Cvpric. Thm \( S_n \xrightarrow{a.s.} \) \( (\sum_{k=1}^{\infty} \frac{\sigma_k^2}{X_k} \) converges a.s.)

\( \therefore \) \( S_n \xrightarrow{L^1} \) by u.i.
(c) If $\sum_{k=1}^{\infty} X_k$ converges a.s., prove that $\sum_{k=1}^{\infty} \sigma_k^2 < \infty$.

**Hint:** If $T_\ell = \min\{n \geq 0 : |S_n| \geq \ell\}$ (min 0 = $\infty$), $M_n$ is as in (a), and $M_{n+1}^{T_\ell} = M_{n+1}^{T_\ell}$, show that

$$\sup_{n \geq 0} |M_{n+1}^{T_\ell} - M_n^{T_\ell}| \leq C_\ell$$

for some finite real constant $C_\ell$.

**Proof:** Let $H_\ell = \sup_n |S_n^{T_\ell}| \leq \ell + B$

$$\sup_n |M_n^{T_\ell}| \leq \sup_n |S_n^{T_\ell}|^2 + \sup_n S_n \leq (\ell + B)^2 + B^2$$

$$\sup_n |M_{n+1}^{T_\ell} - M_n^{T_\ell}| \leq 2(\ell + B)^2 \leq C_\ell$$

By optional stopping, $M_n^{T_\ell}$ is a martingale with uniformly bounded increments. So with probability 1

(i) $M_n^{T_\ell}$ converges a.s. or (iii) $\lim_{n \to \infty} M_n^{T_\ell} = \alpha$ and $\lim_{n \to \infty} M_n^{T_\ell} = -\alpha$

But $\lim_{n \to \infty} M_n^{T_\ell} = \lim_{n \to \infty} |S_n^{T_\ell}|^2 \leq (\ell + B)^2 < \alpha$

Fix an outside a null set so that $M_n^{T_\ell}$ converges a.s. to

and $S_n$ converges a.s. to $\alpha$. For $\ell$ large $T_\ell(w) = T_\ell$ and

$M_n^{T_\ell}(w) = M_n^{T_\ell}$ converges as $n \to \alpha$

$$\frac{1}{n} \sigma_n^2 = S_n^{T_\ell} - M_n^{T_\ell}$$

$$\frac{1}{n} \sigma_n^2 \leq \alpha$$

$$\frac{\alpha^2}{\alpha} \leq \alpha$$