Solutions to Positive Fermat Questions

1. (a) Let $N$ and let $A_j = B(1/n) - B^{1/n}$, $i = 1, \ldots, n$.
   (a) The $A_i$'s are independent RV's (by definition of a H.R. polynomial).
   $A_i$ is not i.i.d., but $\sum A_i$ are i.i.d. RV's.
   $BLA = \sum A_i$.
   $BLA$ is strongly infinitely divisible.

1b) Let $T_0 = \inf_{t < 1} \{ B(t) = \alpha \}$. $T_0 < \infty$ a.s. by $\text{Law of Large Numbers}$

Lemma: Let $T_0 = \inf_{t < 1} \{ B(t) = \alpha \}$. $T_0 < \infty$ a.s. by $\text{Law of Large Numbers}$

Proof: We abuse notation slightly and view $T_0$ as a map from $C(0,1)$, $T_0(\xi) = \inf_{t < 1} \{ \xi(t) = \alpha \}$.

Then on $\{ T_0(B) < \alpha^\beta \}$

(b) $T_0(B) - T_0(B) = T_{B-\alpha}(B + B\alpha T_0 \alpha) - B$

$\text{RHS} = \inf_{t < 1} \{ B(t + \alpha T_0(B))} - B < B - \alpha^\beta$
$\times \inf_{t < 1} \{ B(t + \alpha T_0(B))} = B^{\frac{1}{\beta}} - T_0$
$= \inf_{t < 1} \{ B(t) = \alpha \} - T_0$
$= T_B - T_0$. 

So by the BNP.

$P(T_B - T_0 \leq x) \leq \int_{T_0}^{T_B} P(T_0 \leq x) P(B - \alpha = x) dB$

$P(T_0 \leq x) = P(B - \alpha = x)$

$P(B - \alpha = x) = p_0(B - \alpha)$

$= P(T_0 \leq x) \cdot p_0(B - \alpha)$. (Note $p_0(B - \alpha)$)
The Lemma implies ii) $T_b - T_a$ is ind' by $x_{\tau_0}$, iii) $T_b - T_a \equiv T_{b-a}$.

Let now assume $\delta_j = T_{b-j} - T_j$. Claim $\delta_j$, $i = \delta_j$ are ind' by $\mathbb{m}$.

Induct on $m_0$, $m_0\in \mathbb{N}$ various.

Assume the result for $m_0$. By ii), $\delta_{m_0}$ is ind' by $T_{m_0}$.

For $i = m_0$, $\delta_j = T_{m_0-j} < T_{m_0-j}$. \(\delta_{m_0-j} \) is ind' by $\{\delta_j, \ldots, \delta_m\}$.

Since $\delta_j$, $i = \delta_j$ are ind' by induction, this implies $\delta_j, \ldots, \delta_{m_0-j}$ are ind' by $\mathbb{m}$.

and the induction is complete. So $\delta_j, i = \delta_j$ are ind' since

iii) above implies each $\delta_j$ is equal in law to $T_{m_0}$.

\[ \therefore T_1 = \frac{n}{1!} \delta_j \] implies $T_1$ is strongly infinitely divisible.
2(a) True. Let $\nu(i)$ = # of neighbors of $i$ in $G$. 
$\gamma$ is a stationary initial measure for $X$ (in class) 
as we showed in class.

Since $G$ is connected, $X$ is irreducible.
Here if $\mu$ is any stationary measure such that $\mu = \alpha \gamma$ 
for some $\alpha \geq 0$. Since $\nu(G) = \infty$, this means any stationary 
measure is an $\omega$ measure (or $\infty$). In particular 
$X$ has no stationary distribution.
If $X$ was positive recurrent, then $X$ would have a 
stationary distribution $\nu(i) = \frac{1}{\gamma(i)}$, as shown in class.

$X$ cannot be positive recurrent

i.e. either all states are transient or all states are 
null recurrent

(which is stronger than the assertion above.)

Note: We did show this in class but just saying "proved in class" isn't wise. Note, however, it is perfectly reasonable to 
quote the other theorem used in the above argument.

(b) True.

We showed in class that any $T$ and sequence of $(T_k)$ 
stepping times, converges to an $(T_k)$-stepping time.

Let $T_{m,n} = \max_{m \leq k \leq n} T_k$. 

$S_{T_{m,n}} = b_k$ 
$S_{T_k} = b_k e_k$ 

$T_{m,n}$ is only $T_k$-stepping time.

$T_{m,n} = \lim_{n \to \infty} T_{m,n}$ (a limit) is in class $T_k$-stepping time.

$= \sup_{k \to \infty} T_k$

$\lim_{m \to \infty} T_m = \lim_{m \to \infty} T_{m,n}$ (decreasing limit) is in class $T_k$-stepping time.