1. Let \( \{X_n, n \in \mathbb{N}\} \) be a sequence of independent r.v.’s such that
\[
P(X_n = n^3) = P(X_n = -n^3) = (2n^2)^{-1} \quad \text{and} \quad P(X_n = 0) = 1 - n^{-2}.
\]
If \( S_n = \sum_{j=1}^n X_j \), show that \( \{S_n\} \) is an \((\mathcal{F}_n^X)\)-martingale which converges a.s. but is not \(L^1\) bounded. Hence the natural converse to the Martingale Convergence Theorem is not valid.

2. Let \( \{M_n\} \) be a u.i. \((\mathcal{F}_n)\)-martingale and let \( M_\infty \) be the a.s. limit of \( M_n \) as \( n \to \infty \). Let \( T \in \mathbb{Z}_+ \cup \{\infty\} \) be an \((\mathcal{F}_n)\)-stopping time

(a) Show that
\[
M_T = E(M_\infty|\mathcal{F}_T) \quad \text{a.s.} \quad \text{[Here recall Q. 4 in Assignment 4.]} \]

(b) Show that
\[
E(M_T) = E(M_\infty) \quad \text{and} \quad E(|M_T|) \leq E(|M_\infty|).
\]

3. (Extended Borel-Cantelli). Assume \((\mathcal{F}_n)\) is a filtration and \( A_n \in \mathcal{F}_n \). Show that
\[
\{\omega : \omega \in A_n \ i.o.\} = \{\omega : \sum_{n=1}^\infty P(A_n|\mathcal{F}_{n-1})(\omega) = \infty\} \quad \text{a.s.,}
\]
(that is, these sets differ by a \( P \)-null set).

**Hint:** Consider \( X_n = \sum_{k=1}^n (1_{A_n} - P(A_n|\mathcal{F}_{n-1})) \) and show that \( \limsup_n X_n < \infty \subset \{X_n \text{ convergent}\} \) a.s. For the latter show that if \( T_k = \min\{n : X_n \geq k\} \) then \( \{T_k = \infty\} \subset \{X_n \text{ convergent}\} \) a.s.

In the case when the \( \{A_n\} \) are independent, this reduces to the usual Borel-Cantelli Lemma.

4. Give an example of a submartingale \( \{X_n\} \) such that \( \lim_n X_n = -\infty \) a.s.

**Hint:** One example can be found by considering \( \{Z_j\} \) independent r.v.s where \( Z_j \) can only take on two values: \(-1\) and \( a_j > 0 \), and setting \( S_n = \sum_{j=1}^n Z_j \).

5. Prove that the Doob decomposition of an \((\mathcal{F}_n)\)-submartingale is unique.