Math 419/545

Solutions To HW 5

1. \[
\begin{align*}
1 & \quad P_X((X_1, X_2, \ldots, X_n) \in A) - P_X(X \in A) \\
& = \left| \mathbb{E}_X \left( P_X(X \in A) \right) - \sum_y \Pi(y) \mathbb{P}_Y(X \in A) \right| \quad \text{(EMP)} \\
& = \left| \sum_y \left( P^n((x,y)) - \Pi(y) \right) \mathbb{P}_Y(X \in A) \right| \\
& \leq \sum_y \left| P^n((x,y)) - \Pi(y) \right| \mathbb{P}_Y(X \in A) \\
& \leq \sum_y \left| P^n((x,y)) - \Pi(y) \right| \to 0 \quad \text{as } n \to \infty \quad \text{by the Convergence Thm.}
\end{align*}
\]

As this bound is uniform in $A$, the result follows.

2. (a) Assume $Z$ is transient. Let $i, j \in S$.
\[
\sum_n P^n(i,j)^2 = \sum_n P^n(U(i,j), j, j) = E_{U(i,j)}(N(U(i,j), i, j)) < \infty
\]

Let $(i, j)$ be transient for $\mathbb{P}$.
\[
\lim_{n \to \infty} P^n(i,j) = 0
\]

(b) Assume $Z$ is recurrent and $\exists j \in S : \lim_{m \to \infty} P^m(i,j) = c, c > 0$.
\[
\exists \varepsilon > 0 \text{ and } n_m \to \infty \quad \text{s.t.} \quad \lim_{m \to \infty} P^m(i,j) = d_j \geq \varepsilon > 0.
\]

By Cantor diagonalization (take a further subsequence) we may assume \( \forall k \neq j \lim_{m \to \infty} P^m(i,k) = d_k > 0 \).

Since $\mathbb{P}$ is irreducible, aperiodic and $\mathbb{P}$ is recurrent we may apply the Coupling Thm. (Thm. 5.33 in class) to conclude that \( \forall i \in S \lim_{n \to \infty} \sum_{k \in S} |P^n(i,k) - P^n(i,k)| = 0 \).

\[
\text{It follows from (a) and (b) that } \forall i \in S \lim_{m \to \infty} P^m(i,j) = d_j > 0 \quad \text{and } d_j > 0.
\]
By Feller's Lemma,
\[\sum_k \rho_k p(k, x) \leq \lim_{m \to \infty} \sum_i p^m(i, x) p(i, x) = \lim_{m \to \infty} \rho^m(x, x) = \rho(x, x).\]

Then for any \( m \geq 1 \), we have
\[\sum_k \rho_k p(k, x) \leq \sum_i p^m(i, x) p(i, x) \leq \rho(x, x).\]

But \( \lim_{m \to \infty} \rho^m(x, x) = \rho(x, x) \) and as \( p^m(x, x) \to 1 \) as \( m \to \infty \), we may use DCT to see that
\[\rho(x, x) = \lim_{m \to \infty} \sum_k \rho_k p(k, x) = \sum_k \rho_k p(k, x).
\]

Therefore, \( \rho(x, x) = x \leq p(k, x) \). Since \( x \) is a.s.m.,
\[\sum_k \rho_k p(k, x) \leq \rho(x, x) = x \leq p(k, x) \leq \rho(x, x).
\]

Summing over \( k \), we get
\[\sum_k \rho_k p(k, x) = \sum_k \rho_k p(k, x) = \sum_k \rho_k p(k, x).
\]

But \( \sum_k \rho_k = 1 \), so
\[\rho(x, x) = \sum_k \rho_k p(k, x) = \sum_k \rho_k p(k, x).
\]

As the sums are \( x \), we must have \( x = \rho(x, x) \). Therefore, \( \rho(x, x) = x \) is a.s.m. and is a finite s.m., by (1).

Recall by (2), \( \rho = 0 \), so \( \sum_k \rho_k = 0 \) and so \( \rho = 0 \) is s.d.

Recall we showed in class (p. 22, 4.25) that an irreducible chain has a s.d. iff it is positive recurrent. This is a contradiction (since null recurrent). So we must have \( \lim_{n \to \infty} p^n(i, j) = 0 \) for all \( i, j \).