1. (a) Using the notation from Q. 2 on HW 3, 
$h(x_n) = M_n$ is an $(\pi_n^n)$-martingale under $P_x$, by that result.

(b) $E_n(h(x_n(\pi_n^n)A_n)) \pi_n^n)
= E_x(1(\pi_n^n = n) h(x_{\pi_n^n}) + 1(\pi_n^n = n+1) h(x_{\pi_n^{n+1}})) \pi_n^n

= E_x(1(\pi_n^n = n) h(x_{\pi_n^n}A_n)) + E_x(1(\pi_n^n = n+1) h(x_{\pi_n^{n+1}})) \pi_n^n

(1)
E_x(h(x(\pi_n^n)A_n)) \pi_n^n
= E_x(h(x_{\pi_n^n}A_n)) \pi_n^n

where we use the fact that $1(\pi_n^n = n)$ and $h(x_{\pi_n^n}A_n)$ are $\pi_n^n$-measurable,
and the MP in the last line. If $x \in A^c$, $E_x(h(x_{\pi_n^n}A_n)) = h(x) + Gh(x) = h(x)$
(his harmonic on $A^c$). For $\pi_n^n = n+1$, $x_{\pi_n^n}A_n \in A^c$ and so $E_x(h(x_{\pi_n^n})) = h(x_{\pi_n^n})$.
Use this on the last term in (1) and hence conclude

As $h$ is bounded and $x_{\pi_n^n}A_n$ is $\pi_n^n$-measurable, the result follows.

(c) $h(x_{\pi_n^n}) = E_x(h(x_{\pi_n^n}A_n)) = h(x_{\pi_n^n}A_n)$ and so his bounded.

If $x \in A$, $P_x(\pi_n^n = 1) = 1$ and $h(x) = E_x(h(x_{\pi_n^n}A_n)) = f(x)$.

$h$ is clearly $\pi_n^n$-measurable if $E_x(\phi) = \phi$ for any $\phi \in A^c$.

1. Let $x \in A^c$, $\pi_n^n = n+1$ and so for $\phi = f(x_{\pi_n^n}A_n)$, $\phi(x_{\pi_n^n}, x_{\pi_n^n}) = \phi(x_{\pi_n^n}, x_{\pi_n^n})$.

Let $x \in A^c$, $h(x) = E_x(h(x_{\pi_n^n}A_n)) = E_x(E_x(\phi | A_{\pi_n^n}) I_{\pi_n^n^n}) (by)
= E_x(E_x(\phi | A_{\pi_n^n^n}) I_{\pi_n^n^n}) (by EMP.

= E_x(h(x_{\pi_n^n}))

\phi(x_{\pi_n^n}) = E_x(h(x_{\pi_n^n}A_n)) - h(x_{\pi_n^n}) = h(x_{\pi_n^n})

$h$ is harmonic on $A^c$. 
Next consider uniqueness. Let $g : S \to \mathbb{R}$ be bounded, harmonic, $\omega_n B$, and $g = f$ on $A$. Fix $x \in S$. By (b), $g(x(\omega_n B))$ is on $\mathbb{R}^+$-measure $\omega_n$-almost everywhere and so

$$g(x) = E_2 g(x(\omega_n B)) = E_2 g(x(\omega_n B)) \Rightarrow E_2 g(x(\omega_n B)) = \text{almost everywhere}$$

by LCT and $P_2(\omega_n B) = 1$. Since $g = f$ on $A$, $g(x(\omega_n B)) = f(x(\omega_n B))$ and the above implies

$$g(x) = E_2 f(x(\omega_n B))$$

This proves uniqueness.

2. Let $\lambda$ be a non-zero (eigenvalue of $P$ with eigenvector $f \neq 0$

Let $i \geq 0$. \[|\sum_{j=1}^{N} \pi(i,j)f(j)| = |\lambda f(i)| \]

\[\sum_{j=1}^{N} |\pi(i,j)f(j)| = \sum_{j=1}^{N} |\pi(i,j)f(j)| \leq \sum_{j=1}^{N} |\pi(i,j)f(j)| \]

\[\Rightarrow |\sum_{j=1}^{N} \pi(i,j)f(j)| = |\lambda f(i)| \]

\[\Rightarrow \max_{j=1}^{N} |f(j)| \geq |\lambda| \max_{j=1}^{N} |f(j)| \Rightarrow |\lambda| = 1 \lambda \]

3. (a) If $(\lambda, B) = 1, \theta$, then $P = I$ and $P_{\lambda}(X_n = 1) = \mu(n \theta) \forall n$.

Assume $(\lambda, B) \neq (1, 0)$.

$|\lambda - p| > 0$ \(\Rightarrow\) $\lambda = 1$ or $\lambda = -B$ (distinct).

$
\lambda = 1$ has eigenvector $(1, 1)$; $\lambda = -B$ has eigenvector $(1, -B)$

$G = \begin{pmatrix} 1 & \alpha \\ 1 & -B \end{pmatrix}$, $G^{-1} = \frac{1}{B + \alpha} \begin{pmatrix} B & \alpha \\ -1 & 1 \end{pmatrix}$

$D = G^{-1}PG = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \Rightarrow P = GDG^{-1}$

$p^n = G^nD^nG^{-1} = \frac{1}{\sqrt{\lambda + B}} \begin{pmatrix} \lambda & \alpha \\ 1 & -B \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$(\lambda = 1 - \alpha - B)$
\[ p^n = \frac{1}{\alpha + \beta} \left( \frac{\beta + \alpha^n}{\beta - \alpha^n} \right) \]

\[ p^n_l(x_n = 1) = \mu(1) \left( \frac{\beta + \alpha^n}{\beta - \alpha^n} \right) + (1 - \mu(1)) \left( \frac{\beta}{\beta + \alpha^n} \right) \]

\[ = \frac{1}{\alpha + \beta} \left[ \mu(1) \left( \beta + \alpha^n \right) + (1 - \mu(1)) \left( \beta - \beta^n \right) \right] \]

\[ = \frac{\beta}{\alpha + \beta} \left[ \mu(1) \frac{\alpha^n}{\alpha + \beta} \right] + (1 - \alpha - \beta)^n \left[ \mu(1) - \frac{\beta}{\alpha + \beta} \right] \]

(b) Wlkg \( \alpha + \beta > 0 \)

\[ \lim_{n \to \infty} p^n_l(x_n = 1) = p^n_l(x_n = 1) = \mu(1) \]

So \( \alpha + \beta > 2 \) \( \implies \lim_{n \to \infty} (1 - \alpha - \beta)^n = 0 \) and by (1a)

\[ \lim_{n \to \infty} p^n_l(x_n = 1) = \frac{\beta}{\alpha + \beta} \]

(And so \( \lim_{n \to \infty} p^n_l(x_n = 2) = \frac{\alpha}{\alpha + \beta} \)).

If \( \alpha + \beta = 2 \) by (1a)

\[ p^n_l(x_n = 1) = \frac{\beta}{\alpha + \beta} + (-1)^n \left[ \mu(1) - \frac{\beta}{\alpha + \beta} \right] \]

\[ = \begin{cases} 
\mu(1) & \text{if } n \text{ is even} \\
\frac{2\beta - 2\mu(1)}{\alpha + \beta} & \text{if } n \text{ is odd} 
\end{cases} \]

\[ = \begin{cases} 
\mu(1) & \text{if } n \text{ is even} \\
1 - \mu(1) & \text{if } n \text{ is odd} 
\end{cases} \]

so \( p^n_l(x_n = 1) \) oscillates between \( \mu(1) \) and \( \mu(2) \) as \( x_n \) oscillates deterministically between the 2 states.
4. (a) \[ P^3 = \begin{pmatrix} 0 & 1 & 0 \\ 1-p & 0 & p \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1-p & 0 & p \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1-p & 0 & p \\ 0 & 1 & 0 \\ 1-p & 0 & p \end{pmatrix} \]

\[ P^3 = \begin{pmatrix} 0 & 1 & 0 \\ 1-p & 0 & p \\ 0 & 1 & 0 \end{pmatrix} = P \]

(b) If \( n \geq 1 \), \( p^{3n+1} = p^{3n-2} P^3 = p^{3n-2} P \) by (a)

So by induction \( p^{3n+1} = P \quad \forall n \geq 0 \)

\[ p^{2n+1} = p^{2n-1} \quad \forall n \geq 1 \]

\[ p^n = \begin{cases} p & \text{if } n \text{ is odd} \\ p^2 & \text{if } n \text{ is even} \end{cases} \]

(c) \( P_{\mu}(X_{n+1}) = \mu P_n(1) = \begin{cases} \mu P_{\mu}(1) & n \text{ odd} \\ \mu P_{\mu}(2) & n \text{ even} \end{cases} \)

\[ \mu P_{\mu}(1) = \frac{3}{2} \mu(1) P_{\mu}(1) = \mu(1)(1-p) = \frac{1-p}{2} \]

\[ \mu P_{\mu}(2) = \frac{3}{2} \mu(1) P_{\mu}(2) = \mu(1)(1-p) + \mu(3)(1-p) = \frac{1}{2} (1-p) \]

\[ \therefore P_{\mu}(X_{n+1}) = \frac{1-p}{2} \quad \forall n \in \mathbb{N} \quad (\text{if } n=0 \text{ negl. } \mu(1) = \frac{4}{9} \text{ completely}) \]