1. Consider a Markov chain \( X \) defined on the canonical product space \((S^\mathbb{Z}_+, S^\mathbb{Z}_+\)) with general state space \((S, S)\) and transition probability \( p \). Recall that if \( f \) is a bounded measurable function on \( S \), then \( Gf(x) = \int f(y)p(x, dy) - f(x) = E_x(f(X_1)) - f(x) \). \( G \) is called the generator of \( X \). We say \( h \) is a harmonic function on a set \( D \) in \( S \) if \( Gh(x) = 0 \) for \( x \in D \).

   (a) If \( h \) is a bounded harmonic function on \( S \) prove that \( h(X_n) \) is an \((\mathcal{F}_n^X)\)-martingale with respect to every \( P_x \).

   (b) If \( A \in S \) and \( h \) is a bounded function on \( S \) which is harmonic on \( A^c \), show that \( h(X(n \wedge \tau_A)) \) is an \((\mathcal{F}_n^X)\)-martingale with respect to every \( P_x \). Here \( \tau_A = \inf\{n \geq 0 : X_n \in A\} \leq \infty \).

   (c) Let \( A \in S \) satisfy \( P_x(\tau_A < \infty) = 1 \) for every \( x \in S \). Let \( f : A \to \mathbb{R} \) bounded and measurable. Prove that \( h(x) = E_x(f(X(\tau_A))) \) is the unique bounded function on \( S \) which is harmonic on \( A^c \) and equals \( f \) on \( A \).

   \textbf{Hint:} Consider \( \Phi(X_0, X_1, X_2, \ldots) = f(X_{\tau_A}) \). If \( x \in A^c \), under \( P_x \), what is \( \Phi(X_1, X_2, \ldots) \)?

2. If \( P \) is an \( N \times N \) (\( N \) finite) stochastic matrix, prove that all eigenvalues of \( P \) are in \([-1, 1]\), and so conclude that 1 is the largest eigenvalue of \( P \).

3. Consider the \( \{1, 2\} \)-valued Markov chain \( X_n \) with transition matrix

\[
P = \begin{bmatrix}
1 - \alpha & \alpha \\
\beta & 1 - \beta
\end{bmatrix},
\]

where of course \( \alpha, \beta \in [0, 1] \).

   (a) Diagonalize the transition matrix \( P \) and then find \( P_\mu(X_n = 1) \) for any \( n \) and i.d. \( \mu \).

   (b) If \( 2 > \alpha + \beta \) use the above to find \( \lim_{n \to \infty} P_\mu(X_n = 1) \) for any i.d. \( \mu \). What happens if \( \alpha + \beta = 2 \)?
4. Consider the Markov Chain with state space \{1, 2, 3\} and transition matrix

\[
P = \begin{bmatrix}
0 & 1 & 0 \\
1 - p & 0 & p \\
0 & 1 & 0
\end{bmatrix}
\]

Here \(p \in [0, 1]\).

(a) Verify that \(P = P^3\).

(b) Find \(P^n\) for all \(n \in \mathbb{N}\).

(c) If \(\{X_n\}\) is the Markov Chain with transition matrix \(P\) and initial distribution \(\mu\) satisfying \(\mu(1) = \mu(3) = 1/4\) and \(\mu(2) = 1/2\), find \(P(X_n = 1)\) for all \(n \in \mathbb{Z}_+\).